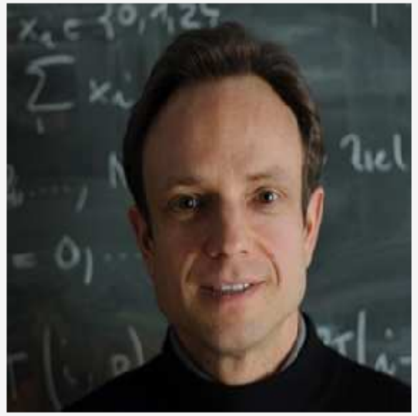
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 1.8: Second-Order Elliptic Variational Problems

Prof. R. Hiptmair, SAM, ETH Zurich

Date: February 12, 2019



Prerequisites.
- Multi-dimensional integration, [Lecture → Section 0.1.2.5] and
- Multi-dimensional integration by parts, [Lecture → Section 1.5.2]

Dependency. Depends on units on [Lecture → Section 1.6] and [Lecture → Section 1.7]

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

I. Second-Order Scalar Elliptic BVPs

1.8. Second-Order Elliptic Variational Problems

Sect. 1.5 : LVP $\overset{\text{IbP}}{\Longrightarrow}$ BVP (strong form)

Now BVP $\Longrightarrow$ (L)VP

Formal procedure : [tacitly assume "sufficient smoothness"]

Formal transition from boundary value problem for PDE to variational problem

STEP 1: *test PDE with smooth functions*
(do not test, where the solution is known, e.g., on the boundary)

STEP 2: *integrate over domain*

STEP 3: *perform integration by parts*
(e.g. by using Green's first formula, Thm. 1.5.2.6)

STEP 4: [optional] *incorporate boundary conditions into boundary terms*

STEP 5: *Choose suitable function spaces (Sobolev spaces)*
(Section 1.3.1: largest function space on which variational problem well posed)

2

Example 1.8.0.2 : Dirichlet BVP

$$-\mathrm{div}(\kappa\,\mathrm{grad}\,u) = f \quad \text{in } \Omega\subset\mathbb{R}^d$$

uniformly positive

$u = g$ on $\partial\Omega$ ← solution known here

Test with $v\in C^\infty(\overline{\Omega})$, $v = 0$ on $\partial\Omega$

Step 1&2:
$$\int_\Omega -\mathrm{div}(\kappa\,\mathrm{grad}\,u)\,v\,dx = \int_\Omega f v\,dx$$

Step 3:
$$\underbrace{\int_\Omega \kappa\,\mathrm{grad}\,u\cdot\mathrm{grad}\,v\,dx}_{=:\,a(u,v)} - \underbrace{\cancel{\int_{\partial\Omega}((\kappa\,\mathrm{grad}\,u)\cdot n)\,v\,dS}}_{=0} = \int_\Omega f v\,dx \quad \forall v\in C_0^\infty(\Omega)$$

Step 4

Step 5: $v\in H_0^1(\Omega)$, $u\in H^1(\Omega)$

Variational form of (1.8.0.3): seek

$$u\in H^1(\Omega),\ u = g \text{ on } \partial\Omega:\quad \int_\Omega \kappa(x)\,\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,dx = \int_\Omega f v\,dx \quad \forall v\in H_0^1(\Omega). \qquad (1.8.0.5)$$

trial space

Example 1.8.0.6 : Radiation BVP

BVP: $\quad -\mathrm{div}(\kappa(x)\,\mathbf{grad}\,u) = f$ in Ω , $\quad -\kappa(x)\,\mathbf{grad}\,u\cdot\boldsymbol{n} = \Psi(u)$ on $\partial\Omega$. (1.8.0.7)

STEP 1 & 2: $u_{|\partial\Omega}$ not fixed $\Rightarrow$ test with $v\in C^\infty(\overline{\Omega})$ ["test everywhere"]

▶ $$-\int_\Omega \mathrm{div}(\kappa(x)\,\mathbf{grad}\,u)\,v\,dx = \int_\Omega f v\,dx \quad \forall v\in C^\infty(\overline{\Omega}).$$

STEP 3 & 4: STEP 3 & 4: apply Green's first formula (1.5.2.7) and incorporate boundary conditions:

▶ $$\int_\Omega \kappa(x)\,\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,dx - \int_{\partial\Omega} \underbrace{\kappa(x)\,\mathbf{grad}\,u\cdot\mathbf{n}}_{=-\Psi(u)\ \text{(STEP 4)}}\,v\,dS = \int_\Omega f v\,dx \quad \forall v\in C^\infty(\overline{\Omega}).$$

Step 5: $v, u\in H^1(\Omega)$ [for "nice" Ψ]

Variational formulation of (1.8.0.7): seek

$$u\in H^1(\Omega):\quad \int_\Omega \kappa(x)\,\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,dx + \int_{\partial\Omega}\Psi(u)\,v\,dS = \int_\Omega f v\,dx \quad \forall v\in H^1(\Omega). \qquad (1.8.0.8)$$

! possibly non-linear V.P.

Special case: impedance BDC [linear !]

$$j\cdot n = q(u-u_0) =: \Psi(u)$$

$$\triangleright\ \underbrace{\int_\Omega \kappa(x)\,\mathrm{grad}\,u\cdot\mathrm{grad}\,v\,dx + \int_\Omega q u\cdot v\,dS}_{=:\,a(u,v)} = \underbrace{\int_\Omega f v\,dx + \int_{\partial\Omega} q u_0 v\,dS}_{=:\,\ell(v)} \quad \forall v\in H^1(\Omega)$$

A LVP!

3 Example 1.8.0.10 : Neumann BVP

BVP:
$$-\operatorname{div}(\kappa(x)\,\mathbf{grad}\,u) = f \quad \text{in } \Omega, \qquad \kappa(x)\,\mathbf{grad}\,u\cdot n = h(x) \quad \text{on } \partial\Omega. \tag{1.8.0.11}$$

↳ uniformly positive

The previous BVP with $\Psi(u) = -h$

Variational formulation of (1.8.0.7): seek

$$u \in H^1(\Omega):\quad \int_\Omega \kappa(x)\,\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,\mathrm{d}x + \int_{\partial\Omega}\Psi(u)\,v\,\mathrm{d}S = \int_\Omega f v\,\mathrm{d}x \quad \forall v \in H^1(\Omega). \tag{1.8.0.8}$$

$\Downarrow$ $\Psi(u) = -h$

$u \in H^1(\Omega)$

(NP) $$\underbrace{\int_\Omega \kappa(x)\,\mathrm{grad}\,u\cdot\mathrm{grad}\,v\,dx}_{=:a(u,v)} = \overbrace{\int_\Omega fv\,dx + \int_{\partial\Omega} hv\,dS}^{=:\ell(v)} \quad \forall v\in H^1(\Omega)$$

- Test with $v \equiv 1$ ▷ balance condition on data

▷ $\int_\Omega f\,dx + \int_{\partial\Omega} h\,dS = 0$

Heat conduction: heat generated inside Ω — heat flowing through $\partial\Omega$

Rem 1.8.0.14: Non-uniqueness of solutions of (NP)

if u solves (NP) $\Rightarrow$ $u+c$ also solves (NP) $\forall c\in\mathbb{R}$

Idea: Impose a constraint to ensure uniqueness

$$u \in H^1_*(\Omega) := \{v\in H^1(\Omega): \int_\Omega v\,dx = 0\} \tag{1.8.0.15}$$

Theorem 1.8.0.20. Second Poincaré-Friedrichs inequality

If $\Omega \subset \mathbb{R}^d$, $d\in\mathbb{N}$, is bounded, then

$$\exists C = C(\Omega) > 0:\quad \|u\|_0 \le C\,\operatorname{diam}(\Omega)\|\mathbf{grad}\,u\|_0 \quad \forall u\in H^1_*(\Omega).$$

▷ $a(u,v)$ is s.p.d. on $H^1_*(\Omega)$

+ norm equivalence: $\|u\|_a \sim |u|_1$

▶ Uniquely solvable variational formulation of Neumann problem:

$$u \in H^1_*(\Omega):\quad \int_\Omega \kappa(x)\,\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x + \int_{\partial\Omega} h v\,\mathrm{d}S \quad \forall v\in H^1_*(\Omega). \tag{1.8.0.16}$$

Review questions 1.8.0.31

Review question(s) 1.8.0.22(Elliptic variational problems)

(Q1.8.0.31.A) What is the meaning and relationship of *classical* and *weak* solutions of 2nd-order elliptic boundary value problems?

(Q1.8.0.31.B) State the linear variational equation and the corresponding quadratic minimization problem related to the second-order linear elliptic boundary value problem

$$-\operatorname{div}\left(\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\mathbf{grad}\, u\right) + u = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \quad , \quad u = 1 \quad \text{on} \quad \partial\Omega\,.$$

(Q1.8.0.31.C) Which linear variational problem gives the weak form of the boundary value problem

$$-\Delta u = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \quad , \quad \begin{aligned} -\mathbf{grad}\, u \cdot \boldsymbol{n} &= u - 1 \quad \text{on} \quad \Gamma_0\,,\\ u &= 0 \qquad\;\; \text{on} \quad \Gamma_1\,,\end{aligned}$$

where $\partial\Omega = \overline{\Gamma}_0 \cup \Gamma_1$ is a partition of the boundary $\partial\Omega$?

(Q1.8.0.31.D) Consider the pure Neumann boundary value problem

$$-\operatorname{div}(\mathbf{A}(\boldsymbol{x})\,\mathbf{grad}\, u) = f \quad \text{in} \quad \Omega \subset \mathbb{R}^d \quad , \quad \mathbf{A}\,\mathbf{grad}\, u \cdot \boldsymbol{n} = h \quad \text{on} \quad \partial\Omega\,.$$

State the compatibility conditions on the data f and h that is necessary for existence of weak solutions. Give a physical interpretation in the context of stationary heat conduction.

(Q1.8.0.31.E) Consider the partial differential equation

$$\mathbf{grad}\,\operatorname{div}\mathbf{u} + c(\boldsymbol{x})\mathbf{u} = \mathbf{f} \quad \text{in}\; \Omega \subset \mathbb{R}^3\,,$$

where $c : \Omega \to \mathbb{R}$ is a bounded and uniformly positive definite coefficient function. Derive the formal variational formulations for boundary value problems for this PDE when equipped with the boundary conditions

1. $\mathbf{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, where $\boldsymbol{n}$ is the exterior unit normal vectorfield on $\partial\Omega$.
2. $\operatorname{div}\mathbf{u} = 0$ on $\partial\Omega$.

△