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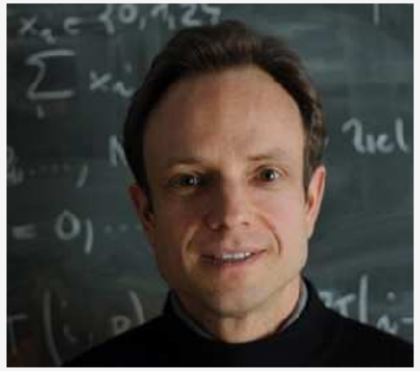
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 1.9: Essential and Natural Boundary Conditions

Prof. R. Hiptmair, SAM, ETH Zurich



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Prerequisites.

- Norms and Cauchy-Schwarz inequality
- Sobolev space, [Lecture → Section 1.3]
- Continuity

Dependency. Builds on [Lecture → Section 1.8], [Lecture → Section 1.7] and [Lecture → Section 1.4]

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

I. Second-Order Scalar Elliptic BVPs

1.9. Essential and Natural Boundary Conditions

☛ 2nd-order elliptic Dirichlet problem:

Strong form : $-\operatorname{div}(\boldsymbol{\alpha}(\mathbf{x})\,\mathbf{grad}\,u) = f$ in Ω , $u = g$ on $\partial\Omega$. (1.5.3.8)

with variational formulation (weak form)

$$\begin{aligned}&u \in H^1(\Omega),\\ &u = g \text{ on } \partial\Omega\end{aligned} : \underbrace{\int_\Omega (\boldsymbol{\alpha}(\mathbf{x})\,\mathbf{grad}\,u(\mathbf{x}))\cdot\mathbf{grad}\,v(\mathbf{x})\,\mathrm{d}\mathbf{x}}_{=a(u,v)} = \int_\Omega f(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \forall v \in H_0^1(\Omega)\,. \qquad (1.4.2.4)$$

affine space

coefficient tensor uniformly s.p.d.

☛ 2nd-order elliptic Neumann problem:

$$-\operatorname{div}(\boldsymbol{\alpha}(\mathbf{x})\,\mathbf{grad}\,u) = f \quad \text{in } \Omega \quad , \quad (\boldsymbol{\alpha}(\mathbf{x})\,\mathbf{grad}\,u)\cdot\mathbf{n} = h \quad \text{on } \partial\Omega\,. \qquad (1.9.0.2)$$

with variational formulation

$$u \in H_*^1(\Omega): \int_\Omega \boldsymbol{\alpha}(\mathbf{x})\,\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,\mathrm{d}\mathbf{x} = \int_\Omega f v\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} h\,v\,\mathrm{d}S \quad \forall v \in H_*^1(\Omega)\,. \qquad (1.8.0.16)$$

vanishing mean

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☛ 2nd-order elliptic mixed Neumann-Dirichlet problem, see Ex. 1.5.3.11:

$$-\operatorname{div}(\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u) = f \quad \text{in } \Omega \;, \qquad \begin{aligned} u &= g \quad \text{on } \Gamma_0 \subset \partial\Omega \;, \\ (\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u)\cdot\boldsymbol{n} &= h \quad \text{on } \partial\Omega\setminus\Gamma_0 \;. \end{aligned} \tag{1.9.0.3}$$

with variational formulation

$$\begin{array}{l} u \in H^1(\Omega)\,, \\ u = g \text{ on } \Gamma_0 \end{array} : \quad \int_\Omega (\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u(\boldsymbol{x}))\cdot\mathbf{grad}\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega\setminus\Gamma_0} h\,v\,\mathrm{d}S \tag{1.9.0.4}$$

for all $v \in H^1(\Omega)$ with $v_{|\Gamma_0} = 0$.

Natural and essential boundary conditions

A pattern has emerged: In the variational formulations of 2nd-order elliptic BVPs of Section 1.8:

Dirichlet boundary conditions are *directly imposed* on trial space and (in homogeneous form) on test space.

Terminology: **essential boundary conditions**

Neumann boundary conditions are enforced *only* through the variational equation.

Terminology: **natural boundary conditions**

Built into the equilibrium principle of the mathematical model

1.9.0.6: What are admissible Dirichlet data $g: \partial\Omega \to \mathbb{R}$?

☛ 2nd-order elliptic Dirichlet problem:

$$-\operatorname{div}(\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u) = f \quad \text{in } \Omega \;, \quad u = g \quad \text{on } \partial\Omega \;. \tag{1.5.3.8}$$

with variational formulation

$$\begin{array}{l} u \in H^1(\Omega)\,, \\ u = g \text{ on } \partial\Omega \end{array} : \quad \int_\Omega (\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u(\boldsymbol{x}))\cdot\mathbf{grad}\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \forall v \in H_0^1(\Omega)\;. \tag{1.4.2.4}$$

Posed on affine space: $V = u_0 + V_0 \subset H^1(\Omega)$

$V_0 = H_0^1(\Omega)$

$u_{0|\partial\Omega} = g$ & $u_0 \in H^1(\Omega)$

▷ g must be the restriction of a function $\in H^1(\Omega)$ to the boundary

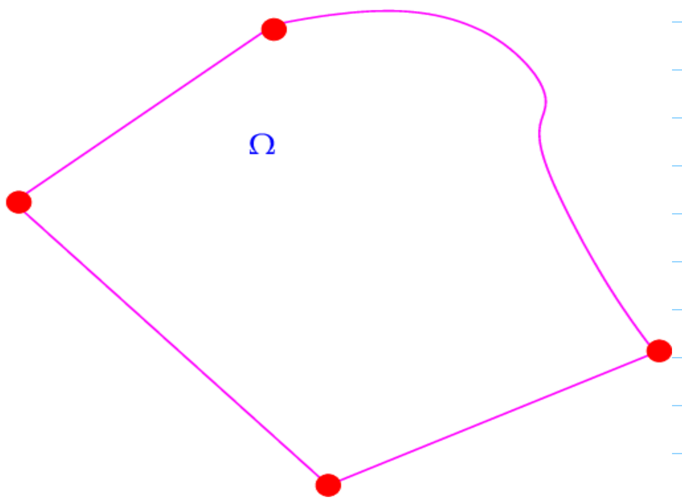
Practical criterion:

✦ Ω is bounded

$$\operatorname{diam}(\Omega) := \sup\{\|\boldsymbol{x}-\boldsymbol{y}\| : \boldsymbol{x},\boldsymbol{y} \in \Omega\} < \infty\,,$$

✦ Ω has piecewise smooth boundary $\partial\Omega$ ▷ ("curvilinear polygon/polyhedron")

For $d = 2$ we can distinguish corners (●) and edges (—) of the boundary $\partial\Omega$.



$$
\begin{array}{ccc}
H^1(\Omega) & \longleftrightarrow & C^1_{pw}(\overline{\Omega}) \subset C^0(\overline{\Omega}) \\
\downarrow & & \downarrow \\
H^{1/2}(\partial\Omega) := H^1(\Omega)|_{\partial\Omega} & \longleftrightarrow & C^1_{pw}(\partial\Omega) \subset C^0(\partial\Omega)
\end{array}
$$

If $g : \partial\Omega \mapsto \mathbb{R}$ is piecewise continuously differentiable (and bounded with bounded piecewise derivatives), , then it can be extended to an $u_0 \in H^1(\Omega)$, *if and only if* it is continuous on $\partial\Omega$.

Important conclusion: $\boxed{\text{Dirichlet boundary values have to be continuous}}$

1.9.0.9: Admissible Neumann data $h : \partial\Omega \to \mathbb{R}$

☛ 2nd-order elliptic Neumann problem:

$$-\operatorname{div}(\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u) = f \quad \text{in } \Omega \quad , \quad (\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u)\cdot\boldsymbol{n} = h \quad \text{on } \partial\Omega\,. \tag{1.9.0.2}$$

with variational formulation

$$u \in H^1_*(\Omega): \quad \underbrace{\int_\Omega \boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u\cdot\mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x}}_{=: a(u,v)} = \underbrace{\int_\Omega f v\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} h v\,\mathrm{d}S}_{=: \ell(v)} \quad \forall v \in H^1_*(\Omega)\,. \tag{1.8.0.16}$$

Remember from Sect. 1.3:

Essential: Continuity of $\ell(\cdot)$ w.r.t. energy norm:

$$|\ell(v)| \leq C\,\|v\|_a \quad \forall v \in H^1_*(\Omega)$$

[α uniformly positive: $\|v\|_a \approx |v|_1$]

Q: For what $h : \partial\Omega \to \mathbb{R}$ holds

$$\exists C > 0: \quad \left|\int_{\partial\Omega} h(x)\,v(x)\,dS(x)\right| \leq C\,|v|_1 \quad \forall v \in H^1_*(\Omega)$$

Key tool: A trace estimate

Theorem 1.9.0.10. Multiplicative trace inequality (MTI)

$$\exists C = C(\Omega) > 0: \quad \underbrace{\|u\|^2_{L^2(\partial\Omega)}}_{= \int_{\partial\Omega} |u(x)|^2\,dS} \leq C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\,.$$

Cauchy-Schwarz inequality in $L^2(\partial\Omega)$

$$\left|\int_{\partial\Omega} h(x)\,v(x)\,dS(x)\right| \leq \|h\|_{L^2(\partial\Omega)} \cdot \|v\|_{L^2(\partial\Omega)}$$

MTI for v

$$\leq C\|h\|_{L^2(\partial\Omega)} \left(\|v\|_0 \cdot \|v\|_1\right)^{1/2}$$

$$\left| \int_{\partial\Omega} h(x)\, v(x)\, dS(x) \right| \leq C \|h\|_{L^2(\partial\Omega)} |v|_1 \qquad (*)$$

2nd Poincaré-Friedrichs: $|v|_0 \leq C |v|_1 \quad \forall v \in H^1_*(\Omega)$

Note: C stands for different constants in different line

$(*) \triangleright$ If $\|h\|_{L^2(\partial\Omega)} < \infty$ then continuity of ℓ

Criterion: $\boxed{h \in L^2(\partial\Omega)}$

Practical criterion: $h \in C^0_{pw}(\partial\Omega)$

$\rightarrow$ Neumann data can be discontinuous

⑤ Review questions 1.9.0.13

Review question(s) 1.9.0.11(Essential and natural boundary conditions)

(Q1.9.0.13.A) For a scalar 2nd-order elliptic boundary value problem for $-\operatorname{div}(\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u) = f$ the Robin boundary conditions read, *cf.* Ex. 1.7.0.5,

$$\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u + \gamma(\boldsymbol{x})u = 0 \quad \text{on } \partial\Omega\,,$$

with $\gamma : \partial\Omega \to \mathbb{R}$ uniformly positive. Are these boundary conditions essential or natural.

(Q1.9.0.13.B) Describe the minimal regularity of Dirichlet and Neumann data for scalar 2nd-order elliptic BVPs on $\Omega \subset \mathbb{R}^d$ in terms of classical smoothness spaces $C^k_{\mathrm{pw}}(\partial\Omega)$.

(Q1.9.0.13.C) Appealing to the following estimate

Theorem 1.9.0.10. Multiplicative trace inequality

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \leq C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\,.$$

prove that for the linear variational problem

$$u \in H^1(\Omega): \quad \int_\Omega \mathbf{grad}\,u \cdot \mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} uv\,\mathrm{d}S = \int_{\partial\Omega} hv\,\mathrm{d}S \quad \forall v \in H^1(\Omega)\,,$$

both the bilinear form and the right-hand side linear form are continuous on $H^1(\Omega)$.

(Q1.9.0.13.D) In the case of the PDE $-\mathbf{grad}\operatorname{div}\mathbf{u} + \mathbf{u} = \mathbf{f}$ on $\Omega \subset \mathbb{R}^3$, what are essential, what are natural boundary conditions. To answer this questions apply Thm. 1.5.2.6 and determine and study the boundary terms arising from it.

(Q1.9.0.13.E) Another stability issue: How does a perturbation (measured in $L^2(\partial\Omega)$-norm) $\delta h \in L^2(\partial\Omega)$ of the Neumann data $h \in L^2(\partial\Omega)$ in the linear variational problem

$$u \in H^1_*(\Omega): \quad \int_\Omega \boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u \cdot \mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x} = \int_\Omega fv\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} h\,v\,\mathrm{d}S \quad \forall v \in H^1_*(\Omega)\,.$$

impact the energy norm of the solution of that variational problem?

△

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