ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 10.1.3: Incompressible Fluids

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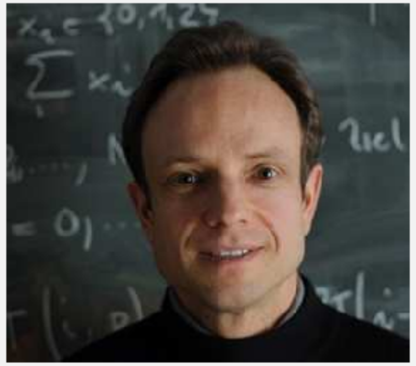

Prerequisites.
- Multi-dimensional integration and differentiation
- Determinants

Dependencies. [Lecture → Section 10.1.1], [Lecture → Section 3.7.1]

Duration: 20 minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

X. Convection-Diffusion Problems

10.1. Heat Conduction in a Fluid

10.1.3. Incompressible Fluids

given: Velocity field $\underline{v}: \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$,
describing motion of fluid

▷ Flow map $(t, \underline{y}) \longmapsto \Phi^t \underline{y}$, $t \in \mathbb{R}$, $\underline{y} \in \Omega$

$$\frac{d}{dt}\Phi^t \underline{y} = \underline{v}(\Phi^t \underline{y})$$

Definition 10.1.3.1. Incompressible flow field

A fluid flow is called **incompressible**, if the associated flow map $\mathbf{\Phi}^t$ is *volume preserving*,

$$|\mathbf{\Phi}^t(V)| = |\mathbf{\Phi}^0(V)| = |V| \quad \text{for all sufficiently small } t > 0 \text{, and for all control volumes } V\,.$$

$$|\Phi^t(V)| = \int_{\Phi^t(V)} 1 \, dx = \int_V |\det D_x \Phi^t(\hat{x})| \, d\hat{x}$$

$$\frac{d}{dt}|\Phi^t(V)| = 0 \iff \frac{\partial}{\partial t}\det D_x \Phi^t(x) = 0 \quad \forall t \;\; \forall x$$

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Theorem 10.1.3.7. Differentiation formula for determinants

Let $\mathbf{S} : I \subset \mathbb{R} \mapsto \mathbb{R}^{n,n}$ *be a smooth matrix-valued function. If* $\mathbf{S}(t_0)$ *is regular for some* $t_0 \in I$, *then*

$$\frac{d}{dt}(\det \circ \mathbf{S})(t_0) = \det(\mathbf{S}(t_0)) \operatorname{tr}\left(\frac{d\mathbf{S}}{dt}(t_0)\mathbf{S}^{-1}(t_0)\right),$$

where $\det : \mathbb{R}^{n,n} \to \mathbb{R}$ *is the matrix determinant and* tr *stands for the trace of a matrix.*

Apply with $S(t) = D_x \phi^t(\underline{x})$, $\underline{x}$ fixed

$$\frac{d}{dt}\{t \longmapsto \det D_x\phi^t(x)\} = \det D_x\phi^t(x) \operatorname{tr}\left(\frac{\partial}{\partial t} D_x \phi^t(x) \cdot \cancel{D_x\phi^t(x)^{-1}}\right)$$

$$\frac{\partial}{\partial t}\phi^t(\underline{x}) = \underline{v}(\phi^t(\underline{x})) \overset{D_x}{\Longrightarrow} \frac{\partial}{\partial t} D_x\phi^t(\underline{x}) \overset{\substack{\text{chain rule}\\ \downarrow}}{=} D\underline{v}(\phi^t(\underline{x}))\, \cancel{D_x\phi^t(\underline{x})}$$

$$\Rightarrow \frac{d}{dt}\{t \longmapsto \det D_x\phi^t(x)\} = \det D_x\phi^t(x) \operatorname{tr} D\underline{v}(\phi^t(x)) \overset{!}{=} 0 \quad \forall t, \underline{x}$$

Note: $\operatorname{tr} D\underline{v}(\underline{y}) = \operatorname{div} \underline{v}(\underline{y}) = 0 \quad \forall \underline{y} \in \Omega$

$$D\underline{v}(\underline{y}) = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & & \frac{\partial v_1}{\partial x_d} \\ & \ddots & \\ \frac{\partial v_d}{\partial x_1} & & \frac{\partial v_d}{\partial x_d} \end{bmatrix}$$

Theorem 10.1.3.8. Divergence-free velocity fields for incompressible flows

A stationary fluid flow in $\Omega \subset \mathbb{R}^d$ *is incompressible* (→ *Def. 10.1.3.1*), *if and only if its associated velocity field* $\mathbf{v} = [v_1, \dots, v_d]^\top : \Omega \to \mathbb{R}^d$ *satisfies* $\operatorname{div} \mathbf{v} = \sum_{j=1}^d \frac{\partial v_j}{\partial x_j} = 0$ *everywhere in* Ω.

§ 10.1.3.9 (Stationary heat equation in an incompressible fluid)

$\operatorname{div} \underline{v} = 0$

Simplification of CDE: ($\rho \equiv$ const)

$$\operatorname{div}(\underline{v}\rho u) = \rho(u \cdot \cancel{\operatorname{div}\underline{v}} + \underline{v} \cdot \operatorname{grad} u) = \rho \underline{v} \cdot \operatorname{grad} u$$

$$-\operatorname{div}(\kappa \operatorname{\mathbf{grad}} u) + \operatorname{div}(\rho \mathbf{v}(\boldsymbol{x}) u) = f \quad \text{in} \quad \Omega$$

← use $\operatorname{div} \mathbf{v} = 0$

$$\boxed{-\kappa \Delta u + \rho \mathbf{v} \cdot \operatorname{\mathbf{grad}} u = f \quad \text{in} \quad \Omega} . \tag{10.1.3.11}$$

Supplement 10.1.3.12 (Maximum principle for convection-diffusion equations)

Remember:

Theorem 3.7.1.2. Maximum principle for 2nd-order elliptic BVP

For $u \in C^0(\overline{\Omega}) \cap H^1(\Omega)$ holds the ***maximum principle***

$$-\operatorname{div}(\kappa(\boldsymbol{x})\,\mathbf{grad}\,u) \geq 0 \implies \min_{\boldsymbol{x}\in\partial\Omega} u(\boldsymbol{x}) = \min_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x})\,,$$
$$-\operatorname{div}(\kappa(\boldsymbol{x})\,\mathbf{grad}\,u) \leq 0 \implies \max_{\boldsymbol{x}\in\partial\Omega} u(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x})\,.$$

Can be generalized:

Theorem 10.1.3.13. Maximum principle for scalar 2nd-order convection diffusion equations $\rightarrow$

Let $\mathbf{v} : \Omega \mapsto \mathbb{R}^d$ be a continuously differentiable vector field and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Then there holds the ***maximum principle***

$$-\Delta u + \mathbf{v}\cdot\mathbf{grad}\,u \geq 0 \implies \min_{\boldsymbol{x}\in\partial\Omega} u(\boldsymbol{x}) = \min_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x})\,,$$
$$-\Delta u + \mathbf{v}\cdot\mathbf{grad}\,u \leq 0 \implies \max_{\boldsymbol{x}\in\partial\Omega} u(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x})\,.$$

heat sources only

▷ Review questions 10.1.3.14

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