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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 10.2.1: Singular Perturbation

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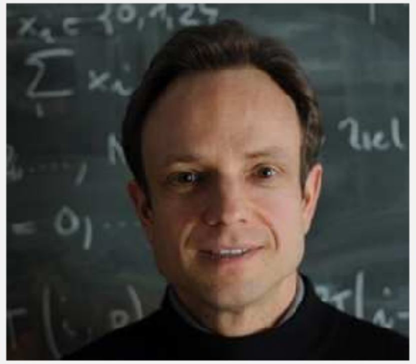
Date: April 27, 2022



Dependencies. [Lecture → Section 10.1.2]

Duration: minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

X. Convection-Diffusion Problems

10.2 Stationary Convection-Diffusion BVPs: Numerics

$\Omega \subset \mathbb{R}^d \triangleq$ bounded computational domain

Model problem:

$$\underbrace{-\epsilon\Delta u}_{\substack{\text{diffusive term} \\ \text{(2nd-order term)}}} + \underbrace{\mathbf{v}(\mathbf{x}) \cdot \mathbf{grad}\, u}_{\substack{\text{convective term} \\ \text{(1st-order term)}}} = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \partial\Omega\,. \tag{10.2.0.1}$$

Assume: $\underline{v} \in C^0(\bar{\Omega})$, $\operatorname{div} \underline{v} = 0$, $\|\underline{v}\|_\infty = 1$

$\hookrightarrow$ incompressible flow

§ 10.2.0.2 (Variational formulation)

$$u \in H_0^1(\Omega):\quad \underbrace{\epsilon\int_\Omega \mathbf{grad}\,u\cdot\mathbf{grad}\,w\,\mathrm{d}x + \int_\Omega (\mathbf{v}\cdot\mathbf{grad}\,u)\,w\,\mathrm{d}x}_{\text{bilinear form } a(u,w)} = \underbrace{\int_\Omega f(x)\,w(x)\,\mathrm{d}x}_{\text{linear form } \ell(w)} \quad \forall w\in H_0^1(\Omega)\,. \tag{10.2.0.3}$$

≙ linear variational problem

- $a(\cdot,\cdot)$ not symmetric $\Rightarrow$ no energy norm
- $a(\cdot,\cdot)$ continuous on $H^1(\Omega)$, since $\mathbf{v}$ bounded
- $w\in H_0^1(\Omega)$: $a(w,w) = \epsilon\int_\Omega \|\mathbf{grad}\,w\|^2\,dx > 0 \quad w\neq 0$

$$\int_\Omega (\mathbf{v}\cdot\mathbf{grad}\,w)\,w\,dx = \int_\Omega (\mathbf{v}w)\cdot\mathbf{grad}\,w\,dx$$

i.b.p.

$$= -\int_\Omega \mathrm{div}(\mathbf{v}\cdot w)\,w\,dx + \int_{\partial\Omega} \underbrace{\cancel{w^2}}_{=0}\,\mathbf{n}\cdot\mathbf{v}\,dS$$

$$= -\int_\Omega \big(w\,\underbrace{\cancel{\mathrm{div}\,\mathbf{v}}}_{=0} + \mathbf{v}\cdot\mathbf{grad}\,w\big)\,w\,dx$$

$= 0$

$\Rightarrow$ $a(\cdot,\cdot)$ is positive definite

$(\epsilon>0)$

▷ Existence and uniqueness of solutions of (10.2.0.3)

10.2.1. Singular Perturbation

$$-\epsilon\Delta u + \mathbf{v}(x)\cdot\mathbf{grad}\,u = f \quad \text{in } \Omega \tag{10.2.0.1}$$
$$u = 0 \quad \text{on } \partial\Omega$$

Fast moving fluids: $\epsilon \ll \|\mathbf{v}\|_\infty = 1$

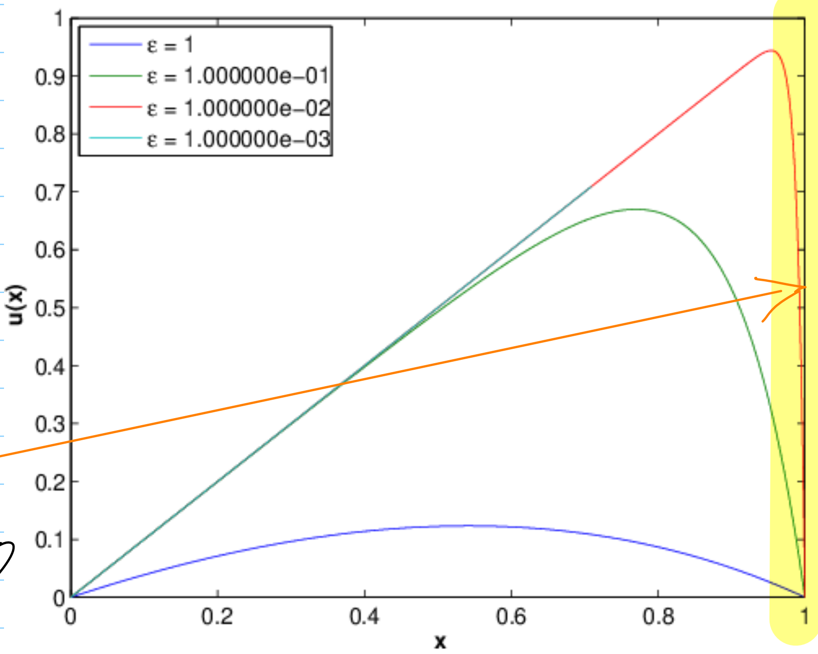
≙ dominant convection

Example 10.2.1.1 (1D CDE)

$d=1$, $v=1$

$$-\epsilon\frac{d^2u_\epsilon}{dx^2} + \frac{du_\epsilon}{dx} = 1 \quad \text{in } \Omega\,, \tag{10.2.1.2}$$
$$u_\epsilon(0) = 0\,,\quad u_\epsilon(1) = 0\,,$$

▶ $$u_\epsilon(x) = x + \frac{\exp(-x/\epsilon)-1}{1-\exp(-1/\epsilon)}\,. \tag{10.2.1.3}$$



$u_\epsilon(x) \to x$ for $\epsilon\to 0$ away from $x=1$

Boundary layer at $x=1$ for $\epsilon\to 0$

$\epsilon>0$: $u_\epsilon(x)\geq 0$: maximum principle

Limit problem $\epsilon:=0$: $\frac{du_0}{dx} = 1 \Rightarrow u_0(x) = x + c\,,\ c\in\mathbb{R}$

③ Limit problem ($\varepsilon = 0$) for general d : pure transport

$$\text{(10.2.0.1)} \quad \overset{\epsilon=0}{\blacktriangleright} \quad \mathbf{v}(\mathbf{x}) \cdot \mathbf{grad}\, u = f(\mathbf{x}) \text{ in } \Omega\,. \tag{10.2.1.4}$$

Case $d = 1$ ($\Omega =]0,1[$, $v = 1$) $\longrightarrow$ Example 10.2.1.1.

$$\text{(10.2.1.4)} \quad \overset{d=1}{\blacktriangleright} \quad \frac{du}{dx}(x) = f(x) \quad \Rightarrow \quad u(x) = \int f\,\mathrm{d}x + C\,. \tag{10.2.1.5}$$

$$u(x) = u(0) + \int_0^x f(s)\,\mathrm{d}s = \int_0^x f(s)\,\mathrm{d}s\,, \tag{10.2.1.6}$$

Problem: b.c. cannot be satisfied in the case $\varepsilon = 0$!

Behavior of $x \longmapsto u_\varepsilon(x)$ for $\varepsilon \to 0$ suggests to keep $u(0) = 0$ and drop $u(1) = 0$.

Rationale: ($d = 1$, $v = 1$, Ex. 10.2.1.1.)

$v > 0 \Rightarrow$ flow from left to right

$\Rightarrow$ information transported from left to right

$\Rightarrow$ u can be prescribed only on the left side

Notion 10.2.1.9. Singularly perturbed problem

A boundary value problem depending on parameter $\epsilon \approx \epsilon_0$ is called singularly perturbed, if the limit problem for $\epsilon \to \epsilon_0$ is not compatible with the boundary conditions.

$d > 1$: $\quad \underline{v}(x) \cdot \operatorname{grad} u = f \quad$ in Ω

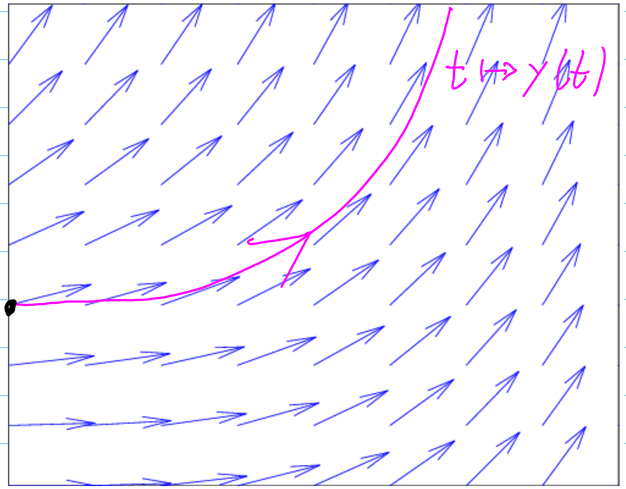
Solution by the method of characteristics :

- $t \longmapsto y(t)$ solves $\dot{y} = v(y)$ [streamline ODE]
- u solves $\operatorname{grad} u \cdot \underline{v}(x) = f(x)$

$$\Rightarrow \frac{d}{dt} u(y(t)) = \operatorname{grad} u(y(t)) \cdot \underbrace{\frac{dy}{dt}(t)}_{\underline{v}(y(t))} = f(y(t))$$

$$\Rightarrow \quad u(y(t)) = u(y(t_0)) + \int_{t_0}^{t} f(y(\tau))\,d\tau \quad \text{(MOC)}$$

choose $y(t_0) \in \partial\Omega$, where u is known



! Follow the flow of information

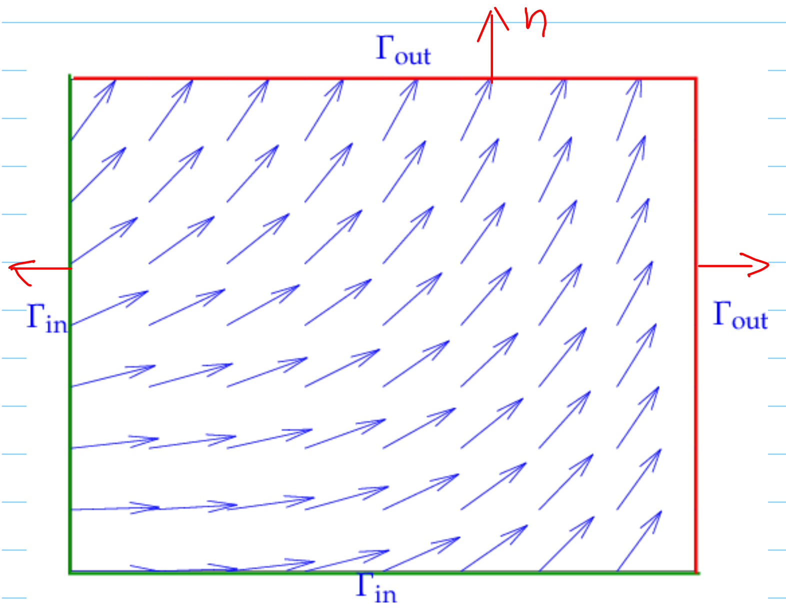
$\varepsilon = 0$: impose b.c. only where flow enters Ω

For the pure transport problem $\mathbf{v} \cdot \mathbf{grad}\, u = f$ Dirichlet boundary conditions can be imposed only on the inflow boundary part

$$\Gamma_{\text{in}} := \{ \boldsymbol{x} \in \partial\Omega : \mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) < 0 \} . \tag{10.2.1.11}$$

but not on its complement in $\partial\Omega$, the outflow boundary part

$$\Gamma_{\text{out}} := \{ \boldsymbol{x} \in \partial\Omega : \mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) > 0 \} . \tag{10.2.1.12}$$



$\varepsilon > 0$: Dirichlet b.c. can be imposed everywhere on $\partial\Omega$

$\varepsilon = 0$: Dirichlet b.c. can be imposed only on Γ_{in}

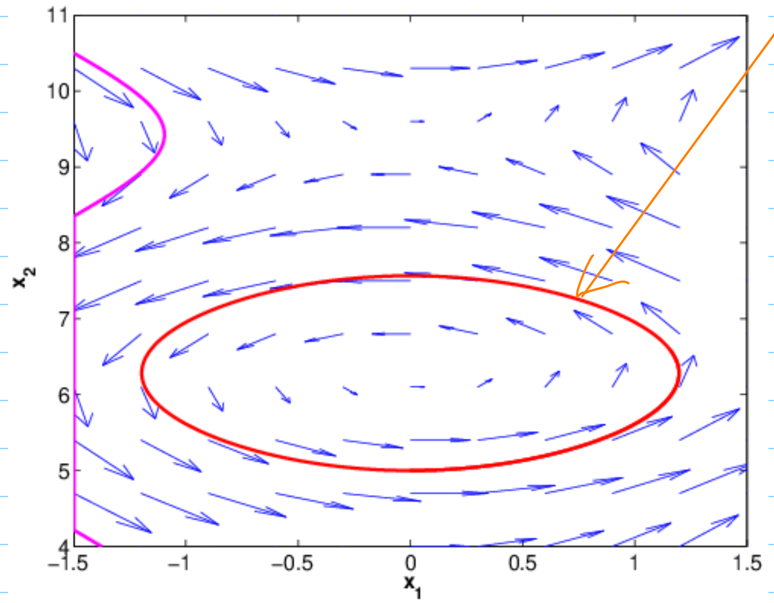
$\Downarrow$

singular perturbation

Notion 10.2.1.9. Singularly perturbed problem

A boundary value problem depending on parameter $\epsilon \approx \epsilon_0$ is called singularly perturbed, if the limit problem for $\epsilon \to \epsilon_0$ is not compatible with the boundary conditions.

Remark 10.2.1.13 (Closed streamlines)



◁ No (unique) solution of pure transport problem

$$v(x) \cdot \operatorname{grad} u = f$$

▷ Review questions 10.2.0.6 & 10.2.1.14