ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 10.2.2.1: Upwind Quadrature

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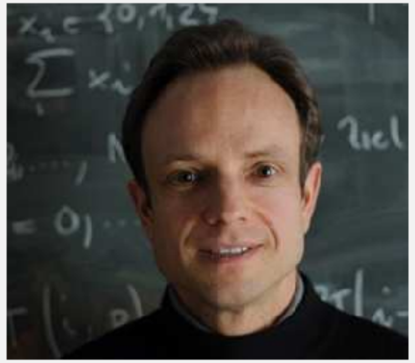

Prerequisites.
- Knowledge about composite quadrature rules

Dependencies. [Lecture → Section 10.1.2], [Lecture → Section 2.4], [Lecture → Section 3.7.2]

Duration: minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

X. Convection-Diffusion Problems

10.2 Stationary Convection-Diffusion BVPs: Numerics

10.2.2. Upwinding

10.2.2.1. Upwind Quadrature

$\Omega \subset \mathbb{R}^d \triangleq$ bounded computational domain

Model problem :

$$\underbrace{-\epsilon\Delta u}_{\substack{\text{diffusive term} \\ \text{(2nd-order term)}}} + \underbrace{\mathbf{v}(\mathbf{x})\cdot\mathbf{grad}\, u}_{\substack{\text{convective term} \\ \text{(1st-order term)}}} = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \partial\Omega\,. \tag{10.2.0.1}$$

singularly perturbed for $\varepsilon \to 0$

[$\varepsilon = 0$: Dirichlet b.c. can only be imposed on inflow boundary]

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Robust discretization in the singular perturbation limit

We desire a **robust discretization** of (10.2.0.1)
= discretization that produces qualitatively correct $(*)$ solutions for **any** $\epsilon > 0$

$(*)$: "qualitatively correct", e.g., satisfaction of maximum principle, Thm. 10.1.3.13]

§ 10.2.2.14 (Upwind quadrature in 1D)

Convection-diffusion 2-pt. BVP; $[v = 1]$

$$-\epsilon\frac{d^2u}{dx^2} + \frac{du}{dx} = f(x) \quad \text{in } \Omega\,, \quad u(0) = 0 \quad , \quad u(1) = 0\,. \tag{10.2.2.1}$$

$$u \in H_0^1(]0,1[): \quad \underbrace{\epsilon\int_0^1 \frac{du}{dx}(x)\frac{dv}{dx}(x)\,dx + \int_0^1 \frac{du}{dx}(x)\,v(x)\,dx}_{=:\mathrm{a}(u,v)} = \underbrace{\int_0^1 f(x)v(x)\,dx}_{=:\ell(v)} \quad \forall v \in H_0^1(]0,1[)\,.$$

- $S^0_{1,0}(\mathcal{M})$ FE Galerkin discretization, meshwidth $h > 0$
- Convective term: num. quad. w/ global composite trapezoidal rule

$$\int_0^1 \psi(x)\,dx \approx h\sum_{j=1}^{M-1}\psi(jh)\,, \quad \text{for} \quad \psi \in C^0(]0,1[),\ \psi(0) = \psi(1) = 0\,,$$

ok, since $w_h \in C^0([0,1])$

$$\int_0^1 \frac{du_h}{dx}(x)w_h(x)\,dx \approx h\sum_{j=1}^{M-1}\frac{du_h}{dx}(jh)\,w_h(hj)\,, \quad w_h \in S^0_{1,0}(\mathcal{M})\,. \tag{10.2.2.15}$$

ambiguous: $u_h \in S^0_1(\mathcal{M}) \Rightarrow \frac{du_h}{dx}$ $\mathcal{M}$-p.w. constant

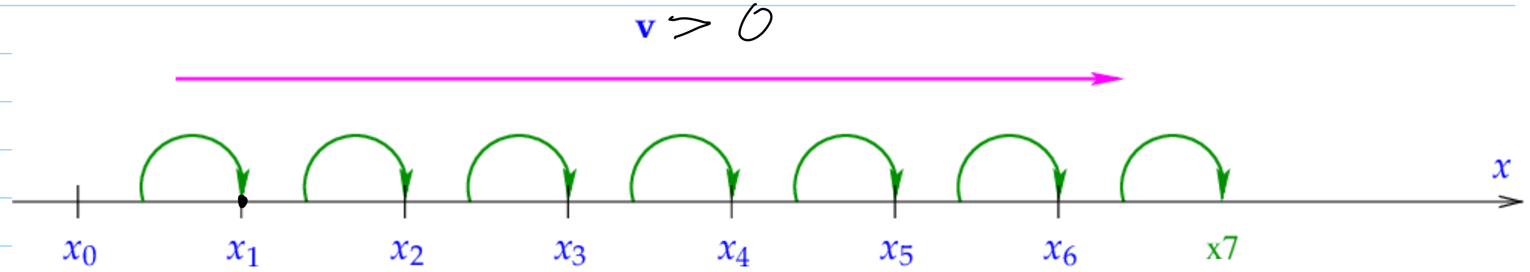
Follow the flow of information:

Idea:
Use upstream/upwind information to evaluate $\frac{du_h}{dx}(jh)$ in (10.2.2.15)

$$\frac{du_h}{dx}(jh) := \lim_{\delta\to 0}\frac{du_h}{dx}(jh-\delta) = \frac{du_h}{dx}_{|]x_{j-1},x_j[}\,.$$

left value

$\hat{=}$ **upwind quadrature**



▷ Tent function basis $\{b_h^1, \dots, b_h^{M-1}\}$:

by elementary properties of b_h^j

$$\int_0^1 \sum_{l=1}^{M-1}\mu_l\frac{db_h^l}{dx}(x)b_h^i(x)\,dx \overset{(10.2.2.15)}{\approx} h\sum_{j=1}^{M-1}\sum_{l=1}^{M-1}\mu_l\frac{db_h^l}{dx}\Big|_{]x_{j-1},x_j[}(hj)\,b_h^i(jh) = h\,\frac{\mu_i - \mu_{i-1}}{h}\,,$$

δ_{ij} a backward d.q.

▶ Linear system from upwind quadrature:

$$\left(-\frac{\epsilon}{h} - 1\right)\mu_{i-1} + \left(\frac{2\epsilon}{h} + 1\right)\mu_i + -\frac{\epsilon}{h}\mu_{i+1} = hf(ih)\,, \quad i = 1,\dots,M-1\,, \tag{10.2.2.8}$$

= LSE obtained by using backward d.q. for $\frac{du}{dx}$!

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§ 10.2.2.16 (Multi-dimensional upwind quadrature)

$\Omega \subset \mathbb{R}^2$, LVP for CD-BVP:

$$u \in H_0^1(\Omega):\quad \underbrace{\epsilon \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, w \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{v} \cdot \mathbf{grad}\, u)\, w \,\mathrm{d}\boldsymbol{x}}_{\text{bilinear form } \mathrm{a}(u,w)} = \underbrace{\int_\Omega f(\boldsymbol{x})\, w(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}}_{\text{linear form } \ell(w)} \quad \forall w \in H_0^1(\Omega)\,. \tag{10.2.0.3}$$

+ $S_{1,0}^0(\mathcal{M})$ Galerkin FEM, $\mathcal{M} \hat{=}$ triangular mesh

I. Use global composite trapezoidal rule

$$\int_\Omega \psi(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \sum_{K\in\mathcal{M}} \int_K \psi(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \approx \sum_{K\in\mathcal{M}} \frac{|K|}{3}\left(\psi(\boldsymbol{a}_K^1) + \psi(\boldsymbol{a}_K^2) + \psi(\boldsymbol{a}_K^3)\right)$$
$$\approx \sum_{\boldsymbol{p}\in\mathcal{V}(\mathcal{M})} \Big(\tfrac{1}{3} \underbrace{\sum_{K\in\mathcal{U}_{\boldsymbol{p}}} |K|}_{\text{area of node patch}}\Big)\, \psi(\boldsymbol{p})\,, \tag{10.2.2.17}$$

for convective part of $\mathrm{a}(\cdot,\cdot)$:

$$\int_\Omega (\mathbf{v} \cdot \mathbf{grad}\, u_h) w_h \,\mathrm{d}\boldsymbol{x} \approx \sum_{\boldsymbol{p}\in\mathcal{V}(\mathcal{M})} \Big(\tfrac{1}{3} \sum_{K\in\mathcal{U}_{\boldsymbol{p}}} |K|\Big) \cdot \mathbf{v}(\boldsymbol{p}) \cdot \mathbf{grad}\, u_h(\boldsymbol{p})\, w_h(\boldsymbol{p})\,. \tag{10.2.2.18}$$

ambiguous for $u_h \in \mathcal{S}_1^0(\mathcal{M})$!

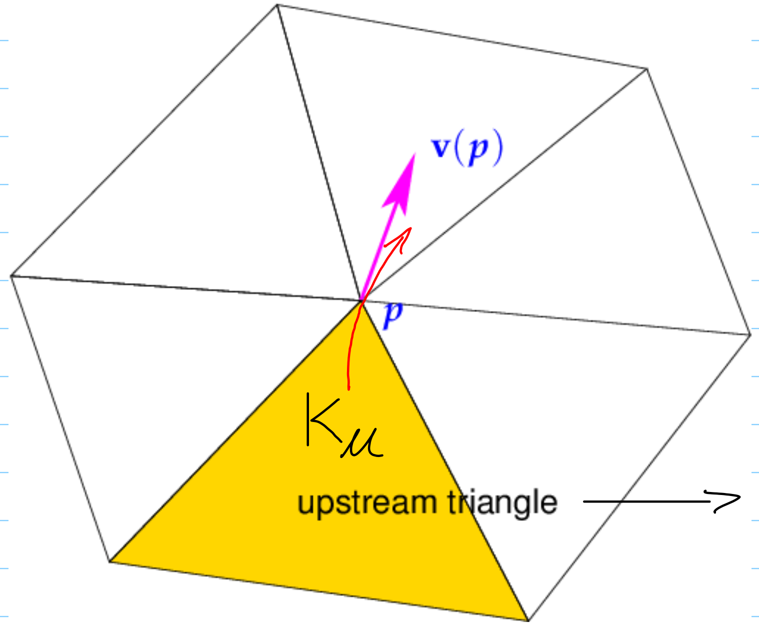
✎ notation: $\mathcal{U}_{\boldsymbol{p}} := \{K \in \mathcal{M} : \boldsymbol{p} \in \overline{K}\}$, $\boldsymbol{p} \in \mathcal{V}(\mathcal{M})$

II. Follow flow of information:

Idea: Use upstream/upwind information to evaluate $\mathbf{grad}\, u_h(\boldsymbol{p})$ in (10.2.2.18)

$$\mathbf{v}(\boldsymbol{p}) \cdot \mathbf{grad}\, u_h(\boldsymbol{p}) := \lim_{\delta\to 0} \mathbf{v}(\boldsymbol{p}) \cdot \mathbf{grad}\, u_h(\boldsymbol{p} - \delta\mathbf{v}(\boldsymbol{p}))\,. \tag{10.2.2.19}$$

$\hat{=}$ general **upwind quadrature**. (UWQ)



upstream triangle $\longrightarrow$ triangle where information comes from

$\mathbf{grad}\, u_h(\boldsymbol{p}) \longleftarrow \mathbf{grad}\, u_{h|K_u}(\boldsymbol{p})$

4 Detailed examination:

Contribution of UWQ convective term to i-th row of FE Galerkin matrix

$$\int_\Omega (\mathbf{v}\cdot\mathbf{grad}\,u_h)w_h\,\mathrm{d}\mathbf{x} \approx \sum_{\boldsymbol{p}\in\mathcal{V}(\mathcal{M})}\Big(\tfrac{1}{3}\sum_{K\in\mathcal{U}_p}|K|\Big)\cdot\mathbf{v}(\boldsymbol{p})\cdot\mathbf{grad}\,u_h(\boldsymbol{p})\,w_h(\boldsymbol{p})\,. \tag{10.2.2.18}$$

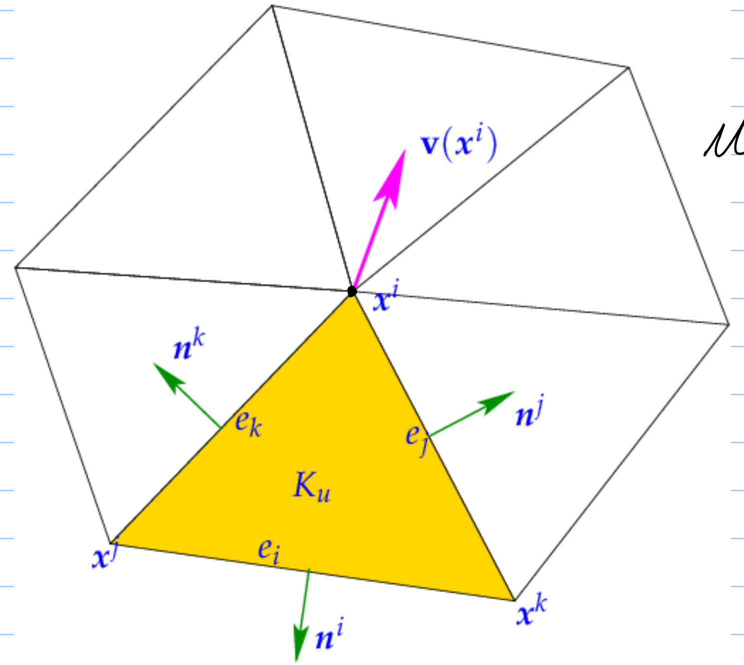
ambiguous for $u_h \in \mathcal{S}_1^0(\mathcal{M})$!

notation: $\mathcal{U}_p := \{K\in\mathcal{M}: \boldsymbol{p}\in\overline{K}\}$, $\boldsymbol{p}\in\mathcal{V}(\mathcal{M})$

$\mathrm{grad}\, u_h|_{K_u}$

$b_h^i \rightarrow w_h$

$$\underbrace{\Big(\tfrac{1}{3}\sum_{K\in\mathcal{U}_i}|K|\Big)}_{=:U_i}\mathbf{v}(\boldsymbol{x}^i)\cdot\mathbf{grad}\,u_{h|K_u}$$



$$u_h|_{K_u} = \mu_j\lambda_j^K + \mu_k\lambda_k^K + \mu_i\lambda_i^K$$

$$\mathrm{grad}\,\lambda_*^K = \frac{|e_*|}{2|K|}\,n^*$$

$* = i, j, k$

▷ i-th row, convective contribution:

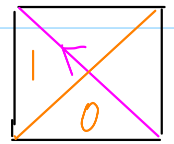
$$\frac{U_i}{2|K_u|}\Big(\underbrace{-\left\|\mathbf{x}^j-\mathbf{x}^k\right\|\mathbf{n}^i\cdot\mathbf{v}(\mathbf{x}^i)\mu_i}_{\leftrightarrow\text{ diagonal entry}} - \left\|\mathbf{x}^i-\mathbf{x}^j\right\|\mathbf{n}^k\cdot\mathbf{v}(\mathbf{x}^i)\mu_k - \left\|\mathbf{x}^i-\mathbf{x}^k\right\|\mathbf{n}^j\cdot\mathbf{v}(\mathbf{x}^i)\mu_j\Big) = \dots$$

< 0 $\quad\geq 0\quad$ ≥ 0

⟺ Sign conditions for discrete maximum principle:

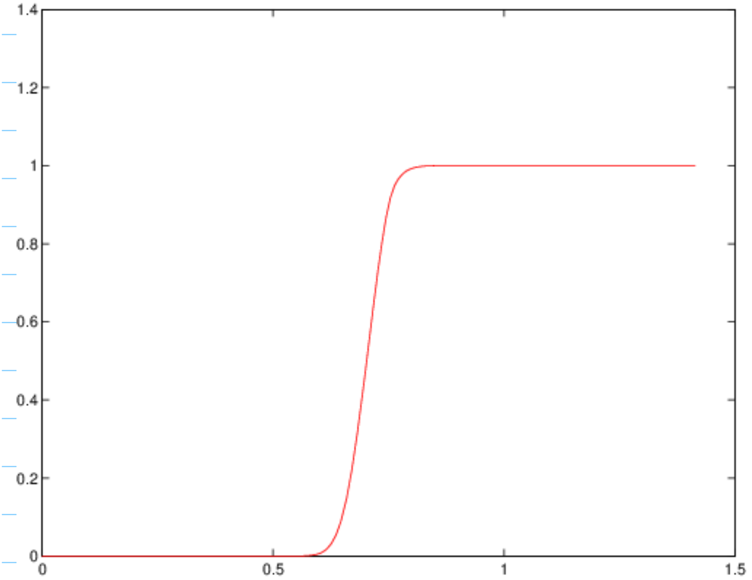
- (3.7.2.8): positive diagonal entries, $\boxed{(\mathbf{A})_{ii} > 0}$,
- (3.7.2.9): non-positive off-diagonal entries, $\boxed{(\mathbf{A})_{ij} \leq 0, \text{ if } i \neq j}$,
- "(3.7.2.10)": diagonal dominance, $\boxed{\sum_j (\mathbf{A})_{ij} \geq 0}$.

Exp. 10.2.2.20 (Upwind quadrature discretization)

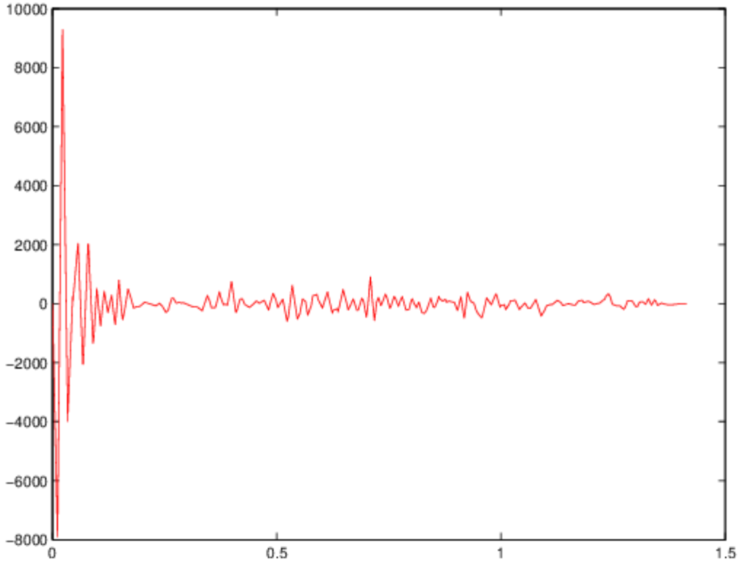


- Computational domain: unit square $\Omega = [0,1]^2$
- $-\epsilon\Delta u + \binom{1}{1}\cdot\mathbf{grad}\,u = 0$
- Dirichlet boundary conditions: $u(x,y) = 1$ for $x > y$ and $u(x,y) = 0$ for $x \leq y$
- Limiting case ($\epsilon \to 0$): $u(x,y) = 1$ for $x > y$ and $u(x,y) = 0$ for $x \leq y$
- layer along the diagonal from $\binom{0}{1}$ to $\binom{1}{0}$ in the limit $\epsilon \to 0$
- 2D triangular Delaunay triangulation, see Rem. 4.2.2.3
- linear finite element upwind quadrature discretization

⑤ $\varepsilon = 10^{-10}$:



upwind quadrature solution



standard Galerkin solution

$\downarrow$

no spurious oscillations

▷ Review questions 10.2.2.21