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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

# Course Video
# Section 10.2.2.2: Streamline Diffusion

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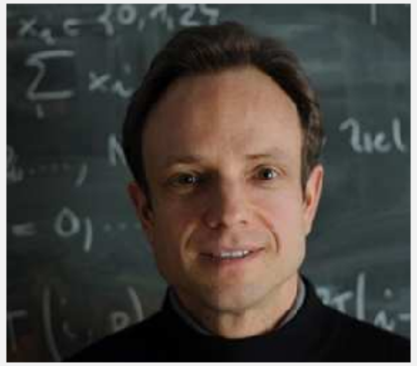

**Dependencies.** [Lecture → Section 10.1.2]

Duration: 35 minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]



# X. Convection-Diffusion Problems

## 10.2. Stationary Convection-Diffusion BVPs: Numerics

### 10.2.2. Upwinding

#### 10.2.2.2. Streamline Diffusion

$\Omega \subset \mathbb{R}^d \triangleq$ bounded computational domain

Model problem:

$$\underbrace{-\epsilon\Delta u}_{\substack{\text{diffusive term}\\\text{(2nd-order term)}}} + \underbrace{\mathbf{v}(\boldsymbol{x})\cdot \mathbf{grad}\, u}_{\substack{\text{convective term}\\\text{(1st-order term)}}} = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \partial\Omega\,. \tag{10.2.0.1}$$

singularly perturbed for $\varepsilon \to 0$

Desirable: $\varepsilon$-robust discretization

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## § 10.2.2.22 (Artificial viscosity in 1D)

1D convection-diffusion model problem, $\Omega = ]0,1[$

$$-\epsilon\frac{d^2u}{dx^2} + \frac{du}{dx} = f(x) \quad \text{in } \Omega\,, \quad u(0)=0 \quad , \quad u(1) = 0\,.$$

Another view of 1D upwinding :

$$\left(-\frac{\epsilon}{h} - 1\right)\mu_{i-1} + \left(\frac{2\epsilon}{h} + 1\right)\mu_i + -\frac{\epsilon}{h}\mu_{i+1} = hf(ih)\,, \quad i = 1,\dots,M-1\,. \tag{10.2.2.8}$$

$$\Updownarrow$$

$$(\epsilon + h/2)\underbrace{\frac{-\mu_{i-1} + 2\mu_i - \mu_{i+1}}{h^2}}_{\hat{=}\ \text{difference quotient for } \frac{d^2u}{dx^2}} + \underbrace{\frac{-\mu_{i-1} + \mu_{i+1}}{2h}}_{\hat{=}\ \text{difference quotient for } \frac{du}{dx}} = f(ih)\,,$$

Upwinding = $h$-dependent enhancement of diffusive term
(This is widely known as **artificial diffusion/viscosity**)

↳ CFD

Extension to $d > 1$ ?

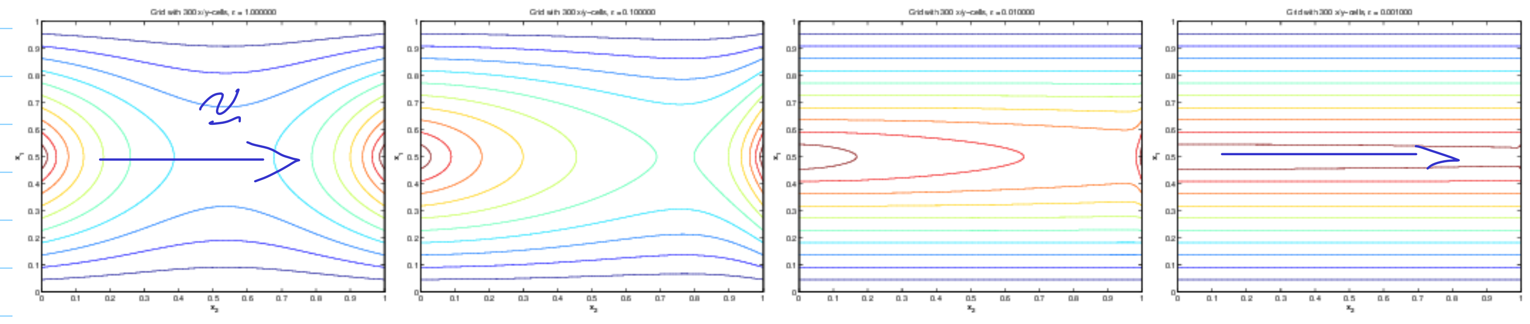
Simply add $h\cdot(-\Delta)$ ?

## Ex. 10.2.2.23 (Effect of added diffusion)

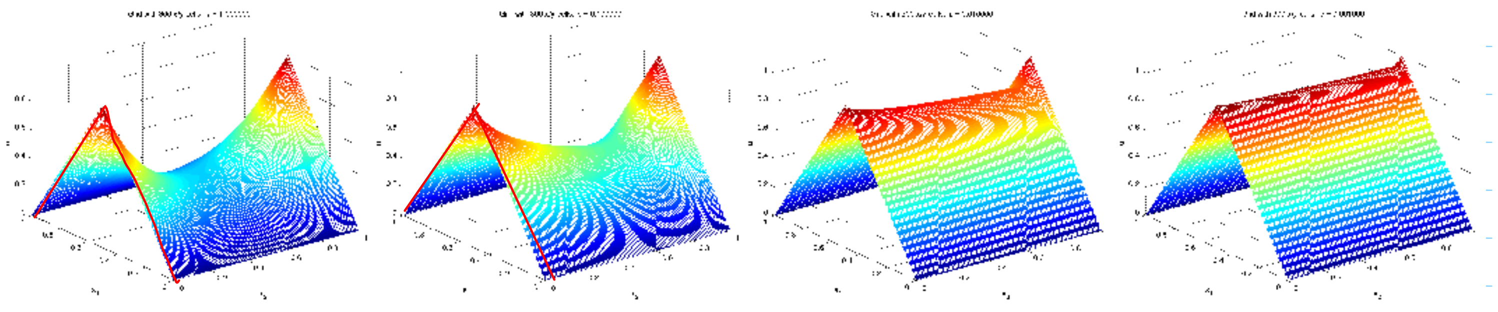
2D CD-BVP:

$\mathbf{v} = \begin{bmatrix}1\\0\end{bmatrix}$ : $\quad -\epsilon\Delta u + \dfrac{\partial u}{\partial x_1} = 0 \quad \text{in } \Omega = ]0,1[^2\,, \quad u = g \quad \text{on } \partial\Omega\,.$

Here, the Dirichlet data are $g(x) = 1 - 2|x_2 - \frac{1}{2}|$ ("roof function").



$\epsilon = 1$ $\quad\epsilon = 0.1$ $\quad\epsilon = 0.01$ $\quad\epsilon = 0.001$



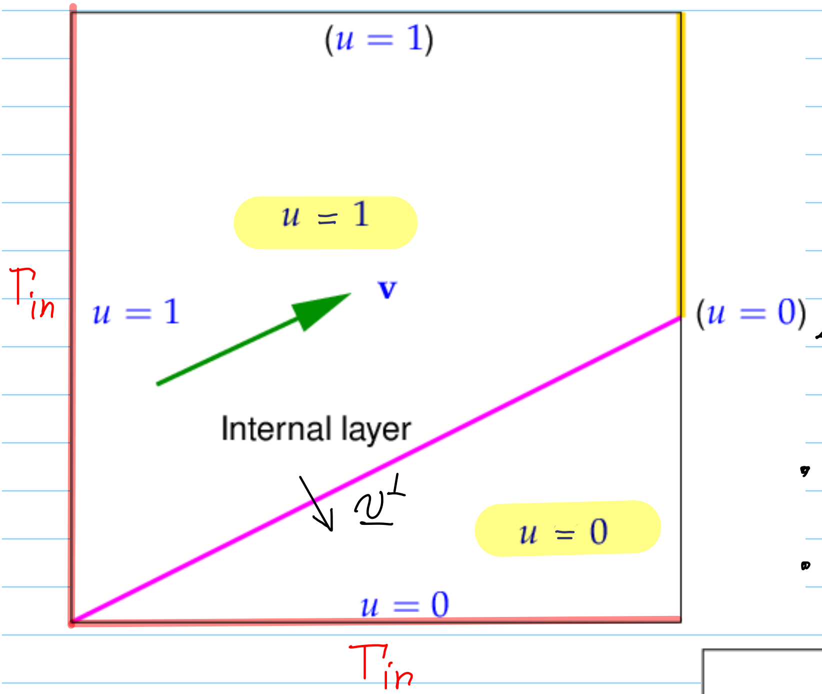
$\epsilon = 1$ $\quad\epsilon = 0.1$ $\quad\epsilon = 0.01$ $\quad\epsilon = 0.001$

[large diffusion] ↓ smooth solution

[tiny diffusion] ↓ sharp internal ridge

Enhancing diffusion ⇒ smearing of internal layers and ridges

## Rem. 10.2.2.24 (Internal layers)




$\varepsilon = 0$ : pure transport

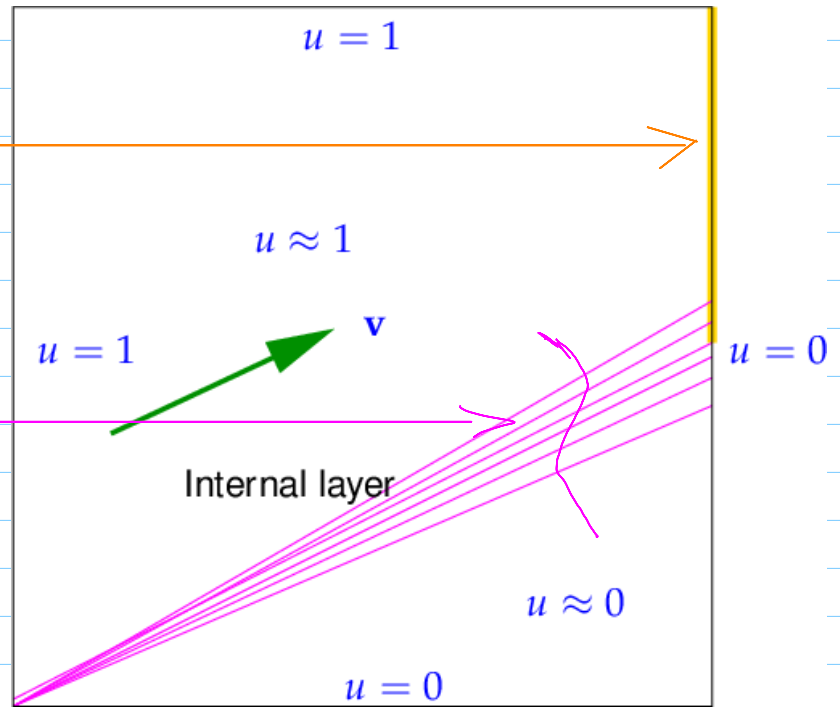
$\underline{v} \cdot \operatorname{grad} u = 0$ in $\Omega$

$\Delta$ Method-of-characteristics solution

- smooth $\parallel$ $\underline{v}$
- jumps $\perp$ $\underline{v}$

$\varepsilon > 0$ :
Boundary layer

Smooth transition instead of sharp jump




(Too much) artificial diffusion ➢ smearing of internal layers

(We are no longer solving the right problem!)

**Heuristics of streamline upwinding** (SU)

Since the solution is *smooth along streamlines*, then adding *diffusion in the direction of streamlines* cannot do much harm.

How to achieve this ?

## § 10.2.2.27 (Anisotropic diffusion)

$\longleftrightarrow$ (Heat conduction) : anisotropic Fourier's law

Heat flux $\underline{j}(\underline{x}) = \underline{v}(\underline{x})\,\underline{v}(\underline{x})^T \operatorname{grad} u(\underline{x})$

$\hat{=}$ diffusion only in the direction of $\underline{v}(\underline{x})$ !

Idea: Anisotropic artificial diffusion in streamline direction

On cell $K$ replace: $\epsilon \leftarrow \underbrace{\epsilon\mathbf{I} + \delta_K \mathbf{v}_K \mathbf{v}_K^T}_{\text{new diffusion tensor}} \in \mathbb{R}^{2,2}$ .

$\mathbf{v}_K \hat{=}$ local velocity (e.g., obtained by averaging)

$\delta_K > 0 \hat{=}$ method parameter controlling the strength of anisotropic diffusion

$\hookrightarrow$ $\varepsilon_K, h_K$ - dependent

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▷ SU-enhanced variational problem

$$\int_\Omega (\epsilon \mathbf{I} + \delta_K \mathbf{v}_K \mathbf{v}_K^T) \,\mathbf{grad}\, u \cdot \mathbf{grad}\, w + \mathbf{v}(\mathbf{x}) \cdot \mathbf{grad}\, u \, w \,\mathrm{d}\mathbf{x} = \int_\Omega f w \,\mathrm{d}\mathbf{x} \quad \forall w \in H_0^1(\Omega)\,. \qquad (10.2.2.30)$$

⚠ This tampering affects the solution $u$ (solution of (10.2.2.30) $\neq$ solution of (10.2.0.1))

Not satisfied yet: We want that the original solution $u$ still solves the stabilized VP!

Goal: consistent modification

**Definition 10.2.2.31. Consistent modifications of variational problems**

A variational problem is called a consistent modification of another, if both possess the same (unique) solution(s).

$$u \in V: \quad a(u,v) = \ell(v) \quad \forall v \in V_0$$

$$\Downarrow$$

Modified LVP: $\quad \tilde{a}(u,v) = \tilde{\ell}(v) \quad \forall v \in V_0$

Idea: Introduce residual (= 0 for exact solution)

$$\int_\Omega \epsilon \,\mathbf{grad}\, u \cdot \mathbf{grad}\, w + (\mathbf{v}(\mathbf{x}) \cdot \mathbf{grad}\, u)\, w \,\mathrm{d}\mathbf{x}$$
$$+ \underbrace{\sum_{K \in \mathcal{M}} \delta_K \int_K (-\epsilon \Delta u + \mathbf{v} \cdot \mathbf{grad}\, u - f) \cdot (\mathbf{v} \cdot \mathbf{grad}\, w) \,\mathrm{d}\mathbf{x}}_{\text{stabilization term}} = \int_\Omega f w \,\mathrm{d}\mathbf{x} \quad \forall w \in H_0^1(\Omega)\,. \qquad (10.2.2.35)$$

stabilization term $= 0$ if $u$ is the exact solution

$$(\mathbf{v} \cdot \mathbf{grad}\, u)(\mathbf{v} \cdot \mathbf{grad}\, w) = (\mathbf{v}\mathbf{v}^T \mathbf{grad}\, u) \cdot \mathbf{grad}\, w$$

→ r.h.s

▷ Exact solution $u$ will also solve (10.2.2.35)

Choice of $\delta_K$: (for p.w. linear FE)

$$\delta_K := \begin{cases} \epsilon^{-1} h_K^2 & \text{, if } \dfrac{\|\mathbf{v}\|_{K,\infty} h_K}{2\epsilon} \leq 1\,, \quad \leftarrow \text{ diffusion dominated} \\ h_K & \text{, if } \dfrac{\|\mathbf{v}\|_{K,\infty} h_K}{2\epsilon} > 1\,. \quad \leftarrow \text{ convection dominated} \end{cases}$$

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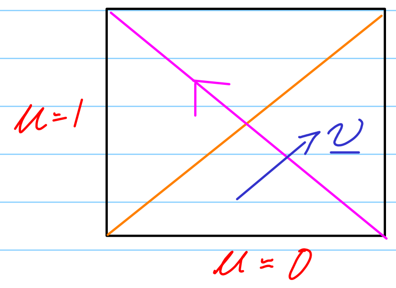
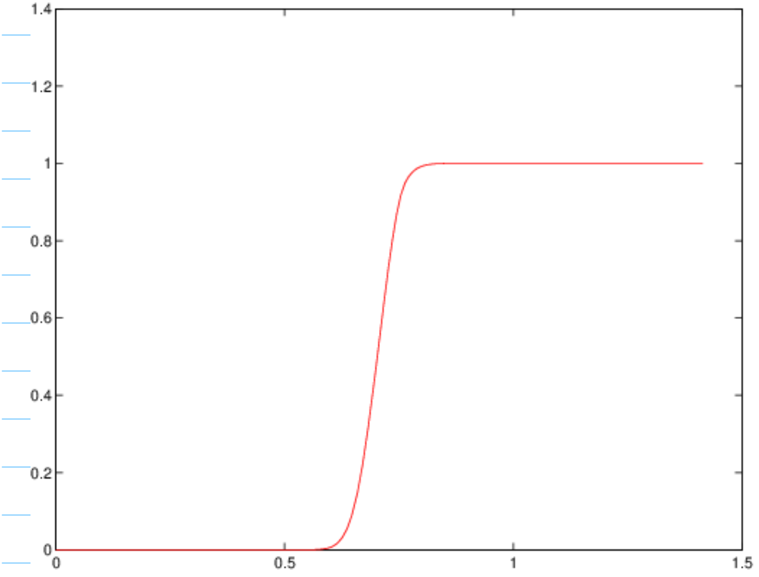
# Exp. 10.2.2.37 (Streamline diffusion discretization: Internal layer)

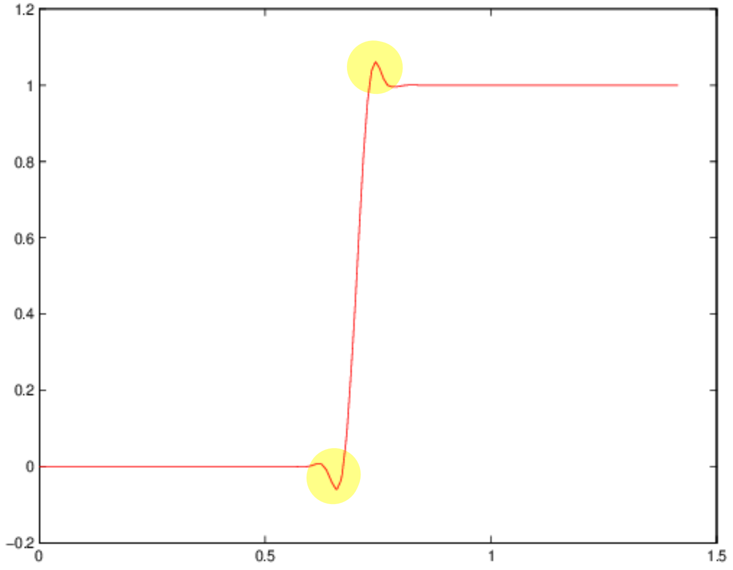
- Computational domain: unit square $\Omega = [0,1]^2$
- $-\epsilon\Delta u + \binom{1}{1} \cdot \mathbf{grad}\, u = 0$
- Dirichlet boundary conditions: $u(x,y) = 1$ for $x > y$ and $u(x,y) = 0$ for $x \leq y$
- Limiting case ($\epsilon \to 0$): $u(x,y) = 1$ for $x > y$ and $u(x,y) = 0$ for $x \leq y$
- layer along the diagonal from $\binom{0}{1}$ to $\binom{1}{0}$ in the limit $\epsilon \to 0$
- 2D triangular Delaunay triangulation, see Rem. 4.2.2.3
- linear finite element upwind quadrature discretization




$\varepsilon = 10^{-10}$ :




upwind quadrature solution




streamline-diffusion solution

sharp jump, but overshoots

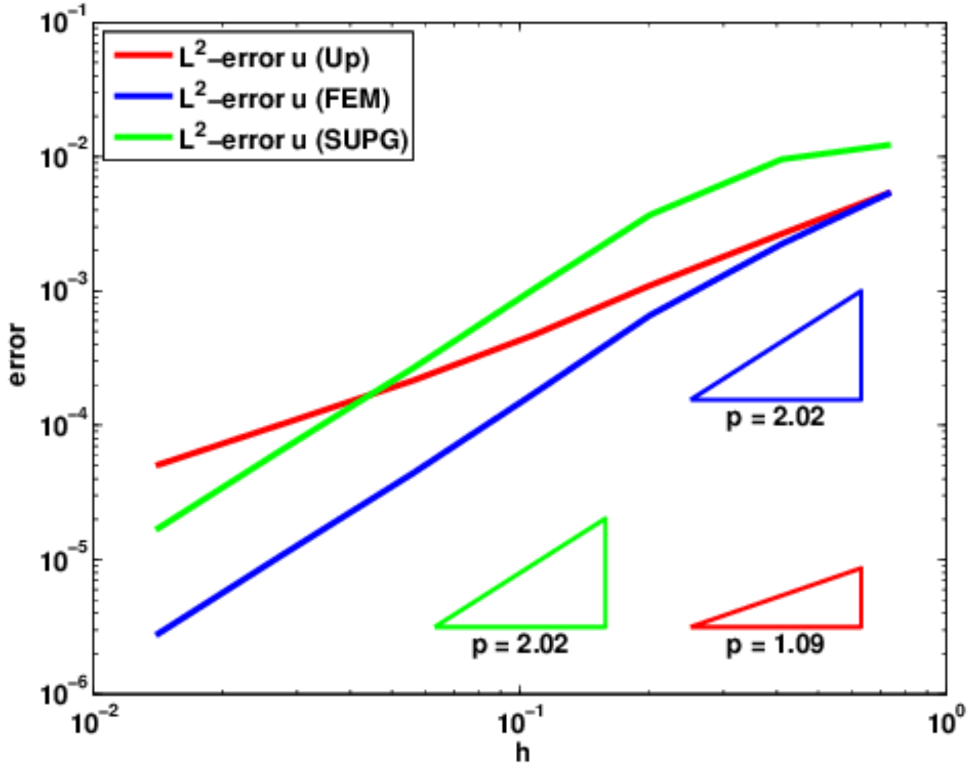
DMP : YES NO

# Exp. 10.2.2.38 (Convergence of SU and UWQ)

- $\Omega = ]0,1[^2$, model problem (10.2.0.1), $\mathbf{v}(\mathbf{x}) = \binom{2}{3}$, right hand side $f$ such that

$$u_\epsilon(x,y) = xy^2 - y^2 e^{2\frac{x-1}{\epsilon}} - x e^{3\frac{y-1}{\epsilon}} + e^{2\frac{x-1}{\epsilon} + 3\frac{y-1}{\epsilon}} .$$ → smooth

- Finite element discretization, $V_{0,h} = \mathcal{S}_1^0(\mathcal{M})$ und sequence of unstructured triangular "uniform" meshes, with
  - upwind quadrature stabilization from Sect. 10.2.2.1,
  - SUPG stabilization according to (10.2.2.35).
- Monitored: (Approximate) $L^2(\Omega)$-norm of discretization error (computed with high-order local quadrature)




Convergence for $\epsilon = 1$ → diffusion dominated

Upwind quadrature
$\|u - u_h\|_{L^2} = O(h)$
for $h \to 0$

Streamline diffusion:
$\|u - u_h\|_{L^2} = O(h^2)$
for $h \to 0$

"Meta theorem":

No $\varepsilon$-robust linear method that respects the maximum principle strictly can be more than 1st-order convergent in $L^2$!

▷ Review questions 10.2.2.39