1

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 10.2.2: Upwinding

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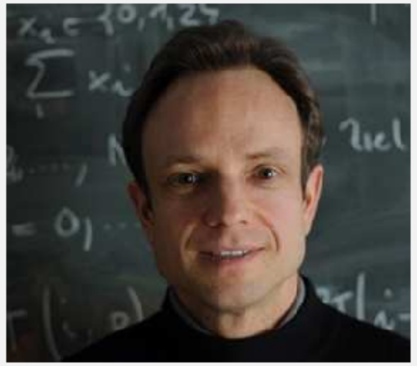


Dependencies. [Lecture → Section 10.2.1], [Lecture → Section 10.1.2], [Lecture → Section 6.2]

Duration: minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

X. Convection-Diffusion Problems

10.2. Stationary Convection-Diffusion BVPs: Numerics

10.2.2. Upwinding

$\Omega \subset \mathbb{R}^d \triangleq$ bounded computational domain

Model problem:

$$\underbrace{-\epsilon\Delta u}_{\substack{\text{diffusive term} \\ \text{(2nd-order term)}}} + \underbrace{\mathbf{v}(\boldsymbol{x}) \cdot \mathbf{grad}\, u}_{\substack{\text{convective term} \\ \text{(1st-order term)}}} = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \partial\Omega\,. \tag{10.2.0.1}$$

singularly perturbed for $\varepsilon \longrightarrow 0$

[$\varepsilon = 0$: Dirichlet b.c. can only be imposed on inflow boundary]

Goal (Dream): Numerical methods that yield accurate solutions *for all* $\varepsilon > 0$.

ε-robust methods

Robust discretization in the singular perturbation limit

We desire a **robust discretization** of (10.2.2.1)

= discretization that produces qualitatively correct $(*)$ solutions for **any** $\epsilon > 0$

$(*)$: "qualitatively correct", e.g., satisfaction of maximum principle, Thm. 10.1.3.13]

Initial focus: Convection-diffusion 2-pt BVP (1D)

$$-\epsilon\frac{d^2u}{dx^2}+\frac{du}{dx}=f(x) \quad \text{in } \Omega\,, \quad u(0)=0 \quad , \quad u(1)=0\,. \tag{10.2.2.1}$$

$$u \in H_0^1(]0,1[): \quad \underbrace{\epsilon\int_0^1 \frac{du}{dx}(x)\frac{dv}{dx}(x)\,\mathrm{d}x + \int_0^1 \frac{du}{dx}(x)\,v(x)\,\mathrm{d}x}_{=:\mathrm{a}(u,v)} = \underbrace{\int_0^1 f(x)v(x)\,\mathrm{d}x}_{=:\ell(v)} \quad \forall v \in H_0^1(]0,1[)\,.$$

Galerkin FE discretization based on $V_{0,h} = S_{1,0}^0(\mathcal{M})$, $\mathcal{M} \mathrel{\hat{=}}$ equidistant mesh of $]0,1[$, mesh width $h > 0$

▷

$$\left(-\frac{\epsilon}{h}-\frac{1}{2}\right)\mu_{i-1}+\frac{2\epsilon}{h}\mu_i+\left(-\frac{\epsilon}{h}+\frac{1}{2}\right)\mu_{i+1}=hf(ih)\,, \quad i=1,\dots,M-1\,, \tag{10.2.2.2}$$

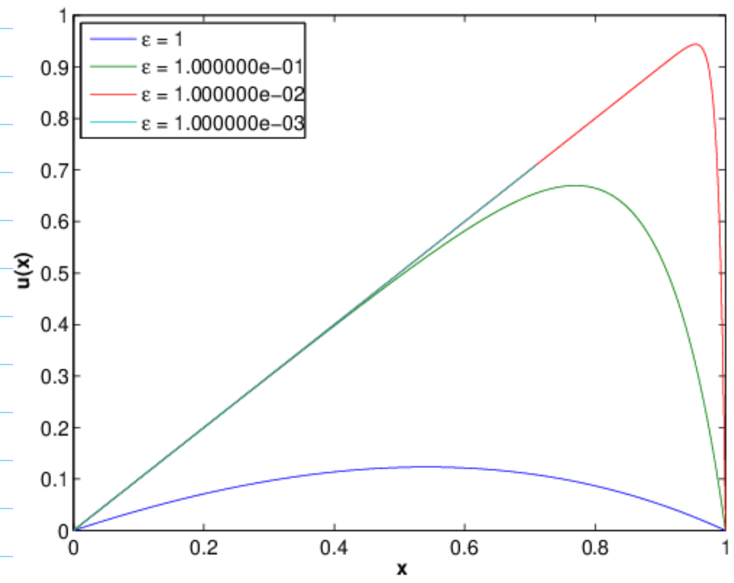
tent-function basis expansion coefficients: $\mu_i = u_h(ih)$

$$-\epsilon\frac{d^2u}{dx^2} \quad + \quad \frac{du}{dx} \quad = \quad f(x)$$

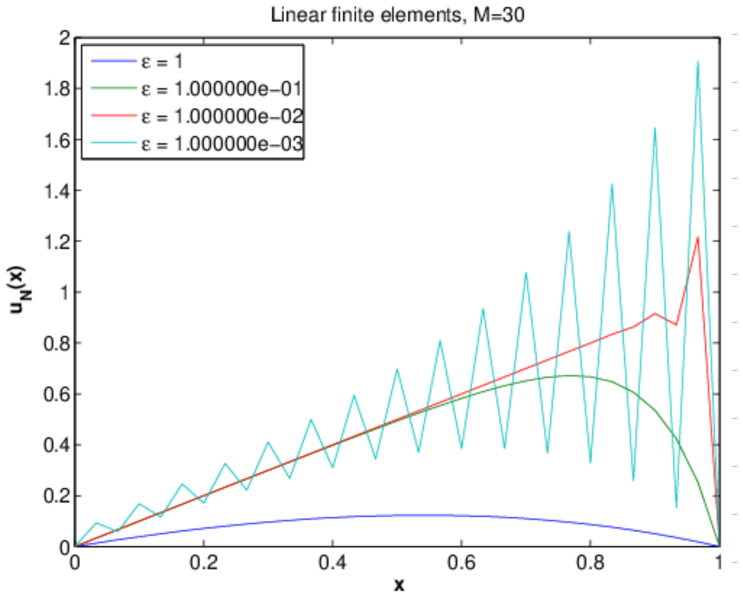
$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow \tag{10.2.2.2}$$

$$\epsilon\underbrace{\frac{-\mu_{i+1}+2\mu_i-\mu_{i-1}}{h^2}}_{\text{difference quotient for } \frac{d^2u}{dx^2}} \quad + \quad \underbrace{\frac{\mu_{i+1}-\mu_{i-1}}{2h}}_{\text{symmetric d.q. for } \frac{du}{dx}} \quad = \quad f(ih)\,.$$

Experiment 10.2.2.4 (Linear FE discretization), $f = 1$



exact solutions



FE solutions

3

Observation : $\varepsilon \ll 1 \Rightarrow$ spurious oscillations

(10.2.2.2) for $\varepsilon = 0$:

$$\mu_{i+1} - \mu_{i-1} = 2h f(ih) \quad , i = 1, \dots, M-1$$

$\rightarrow$ even-odd decoupling (singular LSE for even M)

§ 10.2.2.7 (Discretization of 1D limit problem)

Guideline: Robust numerical methods for singularly perturbed problems must cope with the limit problem

Limit problem, $\varepsilon = 0$: $\frac{du}{dx}(x) = f(x) \;\hat{=}\;$ ODE

Explicit Euler method: $\mu_{i+1} - \mu_i = h f(\xi_i) \quad i = 0, \dots, M-1$,

Implicit Euler method: $\mu_{i+1} - \mu_i = h f(\xi_{i+1}) \quad i = 0, \dots, M-1$.

Explicit Euler: $\frac{du}{dx}(x_i) \approx \frac{u(x_{i+1}) - u(x_i)}{h}$, Implicit Euler: $\frac{du}{dx}(x_i) \approx \frac{u(x_i) - u(x_{i-1})}{h}$.

[forward d.q.] [backward d.q.]

one-sided d.q.

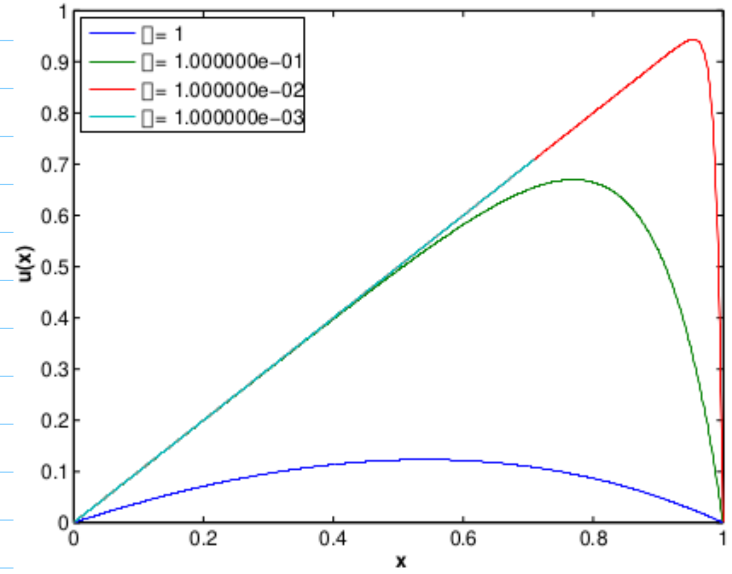
1. Linear system arising from use of *backward difference quotient* $\frac{du}{dx}_{|x=x_i} = \frac{\mu_i - \mu_{i-1}}{h}$:

$$\left(-\frac{\epsilon}{h} - 1\right)\mu_{i-1} + \left(\frac{2\epsilon}{h} + 1\right)\mu_i + -\frac{\epsilon}{h}\mu_{i+1} = hf(ih) \;,\quad i = 1, \dots, M-1 \;, \tag{10.2.2.8}$$

2. Linear system arising from use of *forward difference quotient* $\frac{du}{dx}_{|x=x_i} = \frac{\mu_{i+1} - \mu_i}{h}$:

$$-\frac{\epsilon}{h}\mu_{i-1} + \left(\frac{2\epsilon}{h} - 1\right)\mu_i + \left(-\frac{\epsilon}{h} + 1\right)\mu_{i+1} = hf(ih) \;,\quad i = 1, \dots, M-1 \;, \tag{10.2.2.9}$$

Experiment 10.2.2.10 (One-sided difference approximation of convective terms)



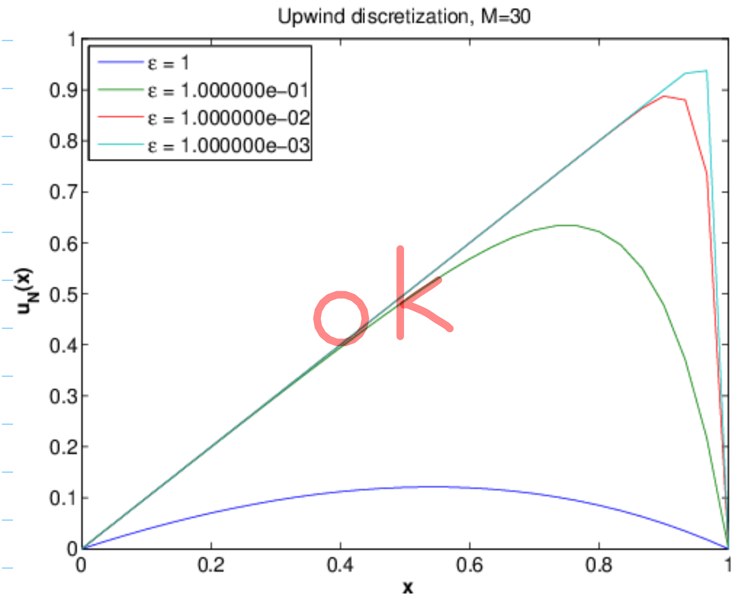
$f = 1$:

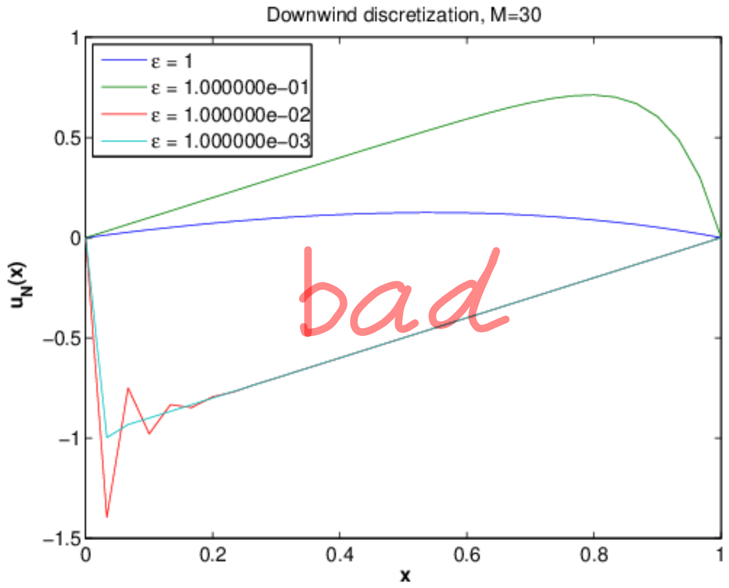
We revisit the model problem of Ex. 10.2.2.4.

◁ exact solutions for different values of ϵ

We study the solutions obtained by the discretizations (10.2.2.8) and (10.2.2.9).

4





backward difference quotient — respects maximum principle

forward difference quotient — violated for small ε

§ 10.2.2.11 (Selection criterion: Discrete maximum principle)

(Linearly interpolated) discrete solution satisfies maximum principle (3.7.2.1).

$\Updownarrow$

System matrix complies with sign-conditions (3.7.2.8)–(3.7.2.10).

- (3.7.2.8): positive diagonal entries, $\boxed{(\mathbf{A})_{ii} > 0}$,
- (3.7.2.9): non-positive off-diagonal entries, $\boxed{(\mathbf{A})_{ij} \leq 0, \text{ if } i \neq j}$,
- "(3.7.2.10)": diagonal dominance, $\boxed{\sum_j (\mathbf{A})_{ij} \geq 0}$.

$\leq 0 \; \forall \varepsilon, h$ $\qquad$ $> 0 \; \forall \varepsilon, h$

1. Linear system arising from *use of backward difference quotient* $\frac{du}{dx}_{|x=x_i} = \frac{\mu_i - \mu_{i-1}}{h}$:

$$\left(-\frac{\epsilon}{h} - 1\right)\mu_{i-1} + \left(\frac{2\epsilon}{h} + 1\right)\mu_i + -\frac{\epsilon}{h}\mu_{i+1} = hf(ih)\,, \quad i = 1, \dots, M-1\,, \tag{10.2.2.8}$$

2. Linear system arising from *use of forward difference quotient* $\frac{du}{dx}_{|x=x_i} = \frac{\mu_{i+1} - \mu_i}{h}$:

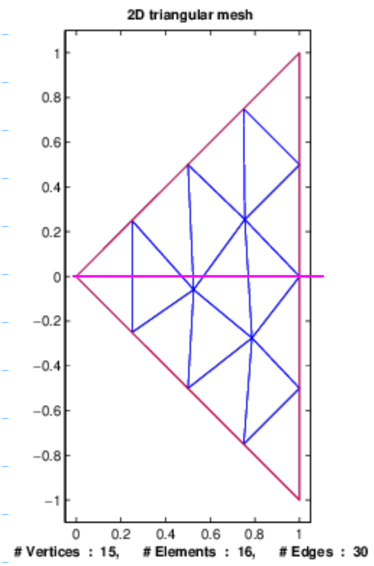
$$-\frac{\epsilon}{h}\mu_{i-1} + \left(\frac{2\epsilon}{h} - 1\right)\mu_i + \left(-\frac{\epsilon}{h} + 1\right)\mu_{i+1} = hf(ih)\,, \quad i = 1, \dots, M-1\,, \tag{10.2.2.9}$$

> 0 only if $h < 2\varepsilon$

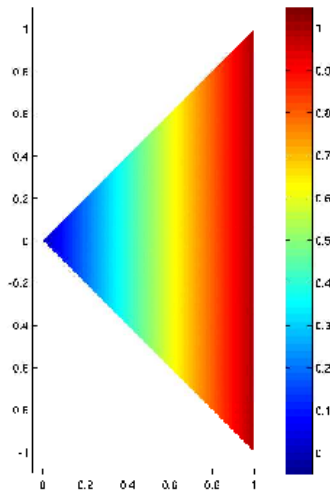
Upwinding

Exp. 10.2.2.12 (Galerkin FEM for CD-BVP in 2D)

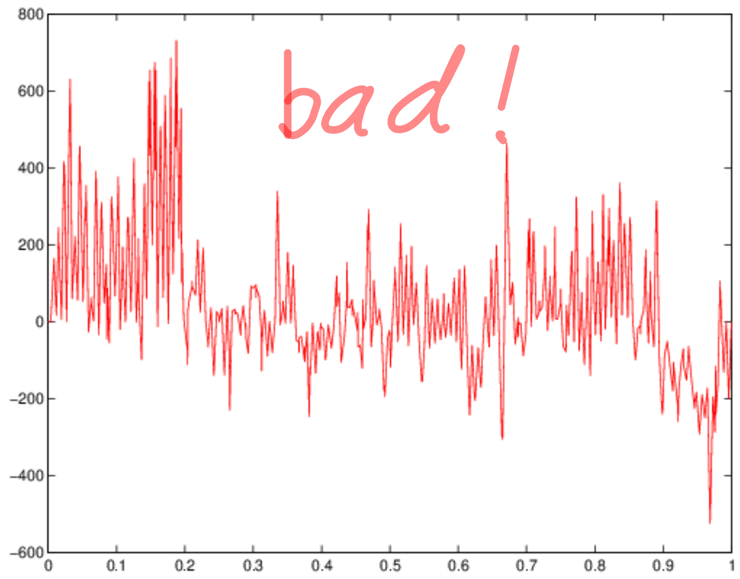
- Triangle domain $\Omega = \{(x,y) : 0 \leq x \leq 1, -x \leq y \leq x\}$.
- Velocity $\mathbf{v}(\boldsymbol{x}) = \binom{1}{0}$ $\succ$ (10.2.0.1) becomes $-\epsilon\Delta u + u_x = 1$.
- Exact solution: $u_\epsilon(x_1, x_2) = x - \frac{1}{1-e^{-1/\epsilon}}(e^{-(1-x_1)/\epsilon} - e^{-1/\epsilon})$, Dirichlet boundary conditions set accordingly
- Standard Galerkin discretization by means of linear finite elements on sequence of triangular mesh created by regular refinement.



Coarse initial mesh



Exact solution ($\epsilon = 10^{-10}$)



Standard Galerkin solution on $x_2 = 0$-line

$\triangleright$ Review questions 10.2.2.13