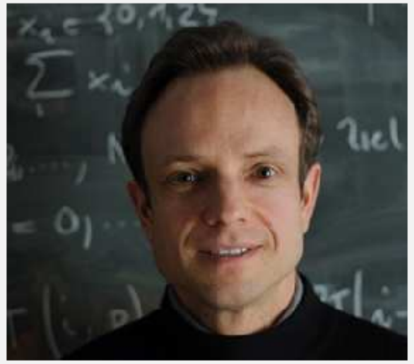
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 10.3.2: Transport Equation



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Prerequisites.

- Basic knowledge about intial-value problems for ODEs

Dependencies. [Lecture → Section 10.1.1], [Lecture → § 10.2.1.7], [Lecture → Section 10.1.4]

Duration: 14 minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

X. Convection-Diffusion Problems

10.3. Discretization of Transient Convection-Diffusion IBVPs

10.3.2. Transport Equation

Focus: Singular perturbation limit $\varepsilon = 0$ for transient convection-diffusion IBVPs:

$$\frac{\partial u}{\partial t} - \epsilon \Delta u + \mathbf{v}(\boldsymbol{x}, t) \cdot \mathbf{grad}\, u = f \quad \text{in } \widetilde{\Omega} := \Omega \times]0, T[\,,$$

$$\longleftarrow \epsilon = 0$$

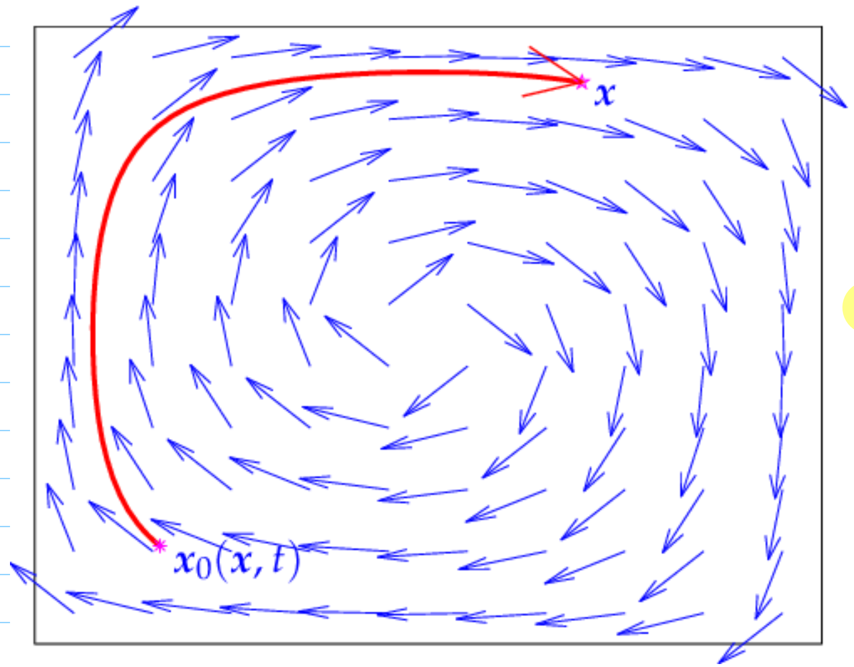
$$\boxed{\frac{\partial u}{\partial t} + \mathbf{v}(\boldsymbol{x}, t) \cdot \mathbf{grad}\, u = f \quad \text{in } \widetilde{\Omega} := \Omega \times]0, T[} \;. \tag{10.3.2.1}$$

Next: Method of characteristics solution formula

→ Sect 10.2.1

Lagrangian view : track behavior of u along trajectories of fluid particles

↳ $t \to y(t)$: $\dot{y}(t) = v(y(t),t)$ (streamline ODE)



u constant on particle trajectories

⇓

$u(x,t) = u_0(x_0(x,t))$

$x_0(x,t) \hat{=}$ starting point of fluid particle in $x \in \Omega$ at t.

$$\frac{d}{dt} u(y(t),t) \overset{\text{chain rule}}{=} \operatorname{grad}_x u(y(t),t) \cdot \dot{y}(t) + \frac{\partial u}{\partial t}(y(t),t)$$

$$= \operatorname{grad}_x u(y(t),t)\, v(y(t),t) + \frac{\partial u}{\partial t}(y(t),t)$$

$$(10.3.2.1) \Rightarrow \quad = f(y(t),t)$$

§ 10.3.2.3 (Solution formula for sourceless transport)

Now : $f = 0 \Rightarrow \frac{d}{dt} u(y(t),t) = 0$

▶ If $f \equiv 0$, then a fluid particle "sees" a constant temperature!

Also assume : $v \cdot n = 0$ on $\partial\Omega$

[No in/outflow]

▷ u by backtracking along streamlines

§ 10.3.2.6 (Method-of-characteristics solution formula)

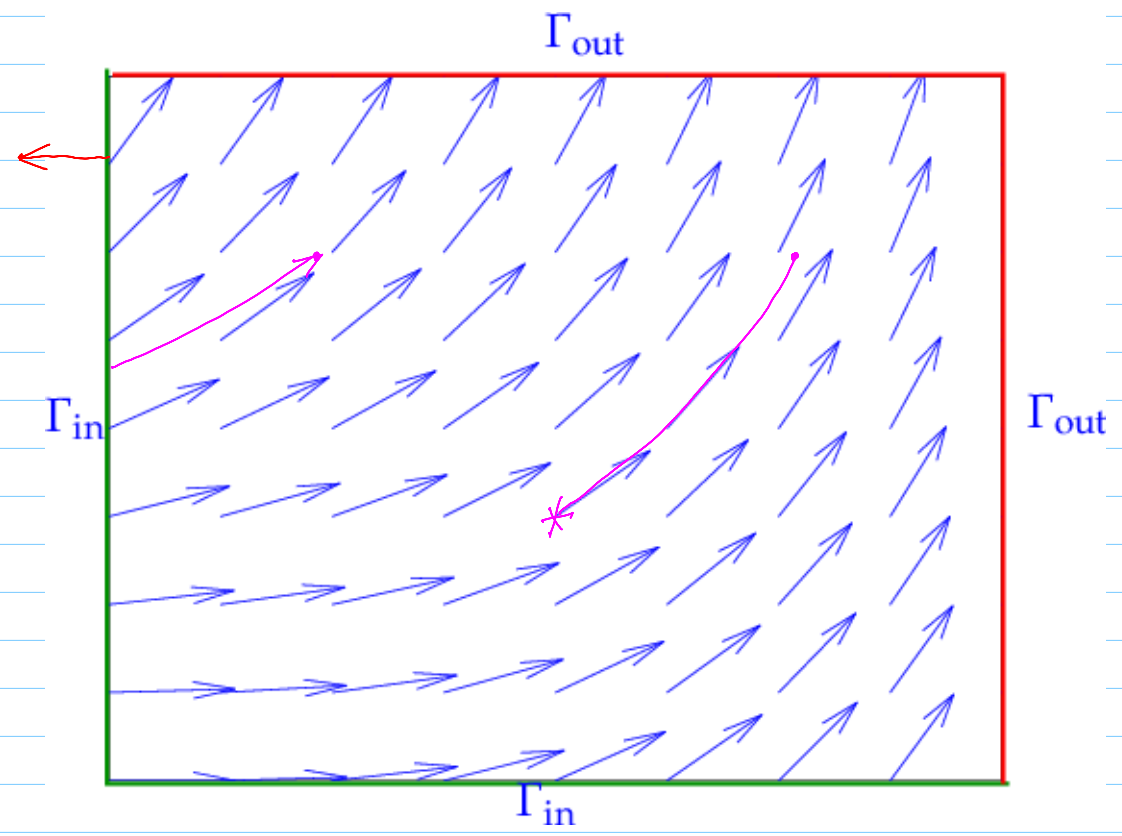
Admit general f, v :

Dirichlet b.c. on inflow boundary :

$u(x,t) = g(x,t)$ for $x \in \Gamma_{in} := \{x \in \partial\Omega : v(x) \cdot n(x) < 0\}$, cf. (10.2.1.11).

↳ where streamlines enter Ω

③



$$\frac{d}{dt}u(\mathbf{y}(t)) = f(\mathbf{y}(t),t) \quad \text{where} \quad \dot{\mathbf{y}}(t) = \mathbf{v}(\mathbf{y}(t),t)\,.$$

▶
$$u(\boldsymbol{x},t) = \begin{cases} u_0(\boldsymbol{x}_0) + \int\limits_0^t f(\mathbf{y}(s),s)\,\mathrm{d}s & \text{, if } \boldsymbol{y}(s)\in\Omega \quad \forall 0<s<t\,, \\ g(\mathbf{y}(s_0),s_0) + \int\limits_{s_0}^t f(\mathbf{y}(s),s)\,\mathrm{d}s & \text{, if } \boldsymbol{y}(s_0)\in\partial\Omega,\ \boldsymbol{y}(s)\in\Omega \quad \forall s_0<s<t\,, \end{cases} \tag{10.3.2.7}$$

streamline starts inside Ω

streamline starts at boundary in $y(s_0)$

▷ Review questions 10.3.2.8

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