ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 10.3.4: Semi-Lagrangian Method

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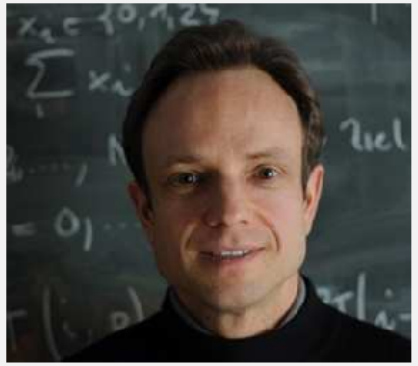
(C) Seminar für Angewandte Mathematik, ETH Zürich



Dependencies. [Lecture → Section 10.3.2]

Duration: 35 minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

X. Convection-Diffusion Problems

10.3. Discretization of Transient Convection-Diffusion IBVPs

10.3.4 Semi-Lagrangian Method

CD-IBVP, Dirichlet b.c.:

$$\frac{\partial u}{\partial t} - \epsilon \Delta u + \mathbf{v}(\boldsymbol{x},t) \cdot \mathbf{grad}\, u = f \quad \text{in} \quad \widetilde{\Omega} := \Omega \times]0,T[\,, \tag{10.3.0.1}$$

- Dirichlet boundary conditions: $u(\boldsymbol{x},t) = g(\boldsymbol{x},t) \quad \forall \boldsymbol{x} \in \partial\Omega\,, \quad 0 < t < T$
- initial conditions: $u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega$.

Particle trajectory $t \mapsto y(t)$ through $(x,t) \in \widetilde{\Omega}$:

$$\dot{y}(t) = v(y(t),t)\,, \quad y(t) = x$$

Rate of change of a function $f:\Omega\times]0,T[\longrightarrow\mathbb{R}$ "sensed" by particle moving with the flow:

Material derivative $\frac{Df}{D\mathbf{v}}(x,t) := \frac{d}{d\tau}\{\tau \longrightarrow f(y(\tau),\tau)\}_{|\tau=t}$

[Chain rule] $= \mathrm{grad}_x f(y(t),t)\cdot\dot{y}(t) + \frac{\partial f}{\partial t}(y(t),t)$

$= \mathrm{grad}_x f(y(t),t)\,\mathbf{v}(y(t)) + \frac{\partial f}{\partial t}(y(t),t)$

$$\boxed{\frac{Df}{D\mathbf{v}}(\boldsymbol{x},t) = \mathbf{grad}_x f(\boldsymbol{x},t)\cdot \mathbf{v}(\boldsymbol{x},t)+\frac{\partial f}{\partial t}(\boldsymbol{x},t)} . \tag{10.3.4.5}$$

§ 10.3.4.4 (Convection-diffusion equation with material derivative)

$$\frac{\partial u}{\partial t} - \epsilon\Delta u + \mathbf{v}(\boldsymbol{x},t)\cdot\mathbf{grad}\, u = f \quad \text{in} \quad \widetilde{\Omega} := \Omega\times]0,T[\,,$$

$$\Updownarrow \;\leftarrow (10.3.4.5)$$

$$\frac{Du}{D\mathbf{v}} - \epsilon\Delta u = f \quad \text{in} \quad \widetilde{\Omega} := \Omega\times]0,T[\,. \tag{10.3.4.6}$$

§ 10.3.4.7 (Semi-Lagrangian temporal semi-discretization)

Idea: Discretize (10.3.4.6) in time by replacing $\frac{Du}{D\mathbf{v}}$ with backward difference quotient

$$\frac{Du}{D\mathbf{v}}(x,t) \approx \frac{u(x,t)-u(y(t-\tau),t-\tau)}{\tau}\,, \quad \text{timestep } \tau>0$$

Rewrite in terms of flow map* Φ: $y(\tau)=\Phi^{t,\tau}x$

* Evolution operator for $\dot{y} = \mathbf{v}(y,t)$: $\frac{\partial\Phi^{t,\tau}x}{\partial\tau} = \mathbf{v}(\Phi^{t,\tau}x)$

↳ Sect. 6.1.4

[Given: temporal mesh $\{0=t_0<t_1<\dots<t_M=T\}$]

▶ This idea yields a *semi-discretization* of (10.3.4.6) *in time* (with fixed timestep $\tau>0$)

$$\frac{u^{(j)}(\boldsymbol{x}) - u^{(j-1)}(\boldsymbol{\Phi}^{t_j,t_j-\tau}\boldsymbol{x})}{\tau} - \epsilon\Delta u^{(j)}(\boldsymbol{x}) = f(\boldsymbol{x},t_j) \quad \text{in} \quad \Omega\,, \tag{10.3.4.9}$$

$+$ initial conditions at $t=t_j$, boundary conditions on $\partial\Omega$,

where $u^{(j)}:\Omega\mapsto\mathbb{R}$ is an approximation for $u(\cdot,t_j)$, $t_j := j\tau$, $j\in\mathbb{N}$.

≙ scalar 2nd-order elliptic BVP for $u^{(j)}$

§ 10.3.4.10 (Finite-element spatial discretization)

Weak form (LVP) of (10.3.4.9) [Zero Dirichlet b.c.]

$$u^{(j)} \in H_0^1(\Omega): \quad \int_\Omega (u^{(j)}(\mathbf{x}) - u^{(j-1)}(\mathbf{\Phi}^{t_j,t_j-\tau}\mathbf{x}))v(\mathbf{x}) + \tau\epsilon\, \mathbf{grad}\, u^{(j)} \cdot \mathbf{grad}\, v \,\mathrm{d}\mathbf{x}$$

$\uparrow$ known

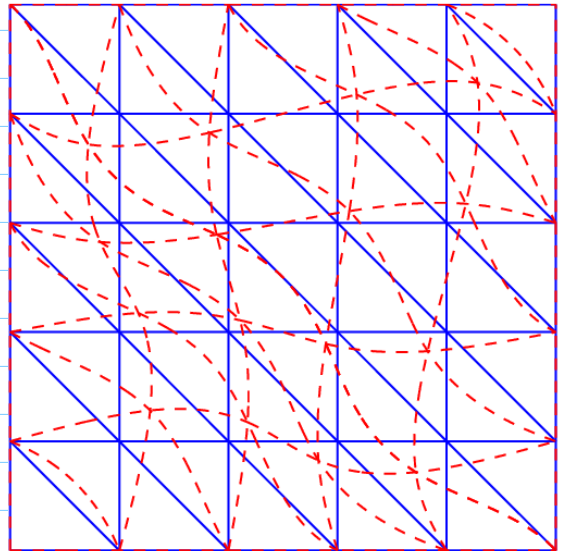
$$= \tau \int_\Omega f(\mathbf{x}, t_j)v(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \forall v \in H_0^1(\Omega)\,, \quad (10.3.4.11)$$

Next: FE Galerkin discretization on a fixed mesh $\mathcal{M}$

$$u_h^{(j)} \in \mathcal{S}_{1,0}^0(\mathcal{M}): \quad \int_\Omega \frac{u_h^{(j)}(\mathbf{x}) - u_h^{(j-1)}(\mathbf{\Phi}^{t_j,t_j-\tau}\mathbf{x})}{\tau} v_h(\mathbf{x})\,\mathrm{d}\mathbf{x} + \epsilon \int_\Omega \mathbf{grad}\, u_h^{(j)} \cdot \mathbf{grad}\, v_h \,\mathrm{d}\mathbf{x}$$

$$= \int_\Omega f(\mathbf{x}, t_j)v_h(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \forall v_h \in \mathcal{S}_{1,0}^0(\mathcal{M})\,. \quad (10.3.4.12)$$

$\notin S_1^0(\mathcal{M})$, not $\mathcal{M}$-p.w. smooth
(even if $\underline{x} \longmapsto \Phi^{t_j,t_j-\tau}\underline{x}$ is smooth)



◁ - - - ≙ image of $\mathcal{M}$ (—) under the mapping $\mathbf{\Phi}^{t_j,t_j-\tau}$

► In general the transported function $\mathbf{x} \mapsto v_h(\mathbf{\Phi}^{t_j,t_j-\tau}\mathbf{x})$ will **not** be a finite element function on $\mathcal{M}$,

▷ No analytic formula for integrals

▷ $\mathcal{M}$-local quadrature problematic

Two more approximations:

1) Replace $\underline{x} \longmapsto u_h^{(j-1)}(\Phi^{t_j,t_j-\tau}\underline{x})$ with its $\mathcal{M}$-p.w. linear interpolant

2) Explicit Euler approximation of $\Phi^{t_j,t_j-\tau}$:

$$\Phi^{t_j,t_j-\tau}\underline{x} = \underline{x} - \underline{v}(x, t_j)\tau$$

▷ Practical semi-Lagrangian method:

$$u_h^{(j)} \in \mathcal{S}_{1,0}^0(\mathcal{M}): \quad \int_\Omega \frac{u_h^{(j)}(\mathbf{x}) - \mathsf{I}_1(u_h^{(j-1)}(\cdot - \tau\mathbf{v}(\cdot, t_j)))(\mathbf{x})}{\tau} w_h(\mathbf{x})\,\mathrm{d}\mathbf{x} + \epsilon \int_\Omega \mathbf{grad}\, u_h^{(j)} \cdot \mathbf{grad}\, w_h \,\mathrm{d}\mathbf{x}$$

$$= \int_\Omega f(\mathbf{x}, t_j)w_h(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \forall w_h \in \mathcal{S}_{1,0}^0(\mathcal{M})\,.$$

▷ LSE for $\vec{\mu}^{(j)} \longleftrightarrow u_h^{(j)} \approx u(\cdot, t_j)$

$$\mathcal{M}(\vec{\mu}^{(j)} - \vec{\eta}(u_h^{j})) + \tau\epsilon A \vec{\mu}^{j} = \tau \vec{\varphi}^{j}$$

$\uparrow$ mass matrix $\qquad$ $\uparrow$ FE-Galerkin matrix for $-\Delta$

$$\vec{\eta}(u_h^{(j-1)}) \longleftrightarrow I_1(u_h^{(j-1)} - \tau\underline{v}(\cdot, t_j))$$

$$\eta_i(u_h^{(j-1)}) = u_h^{(j-1)}(x_i - v(x_i, t_j)\tau)\,, \quad i = 1, \dots, N$$

$\{x_i\}$ ≙ interior mesh nodes

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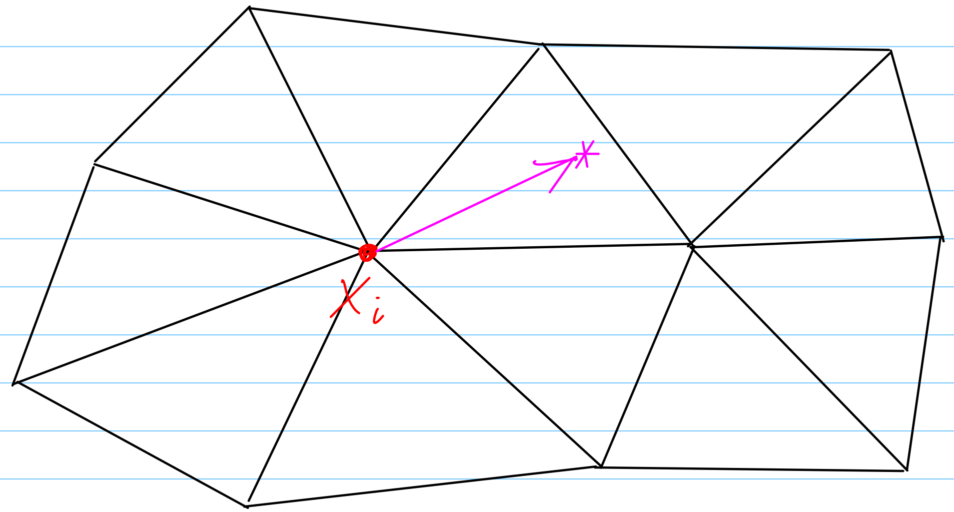
$$\Rightarrow \quad (M + \tau\varepsilon A)\vec{\mu}^{(j)} = M\vec{\eta}(u_h^{(j-1)}) + \tau\vec{\varphi}^{(j)}$$

$$\eta_i(u_h^{(j-1)}) = u_h^{(j-1)}(\underbrace{x_i - v(x_i,t_j)\tau}_{*}) :$$



For small τ : $* \in$ cell adjacent to x_i

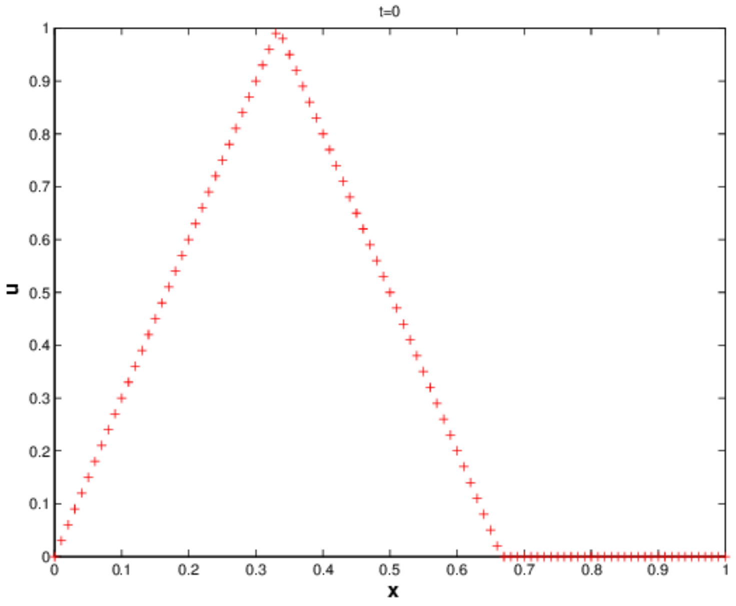
$\Rightarrow$ efficient local search

Exp. 10.3.4.16 (Semi-Lagrangian method for CD-IBVP in 1D)

Same IBVP as in Ex. 10.3.3.20

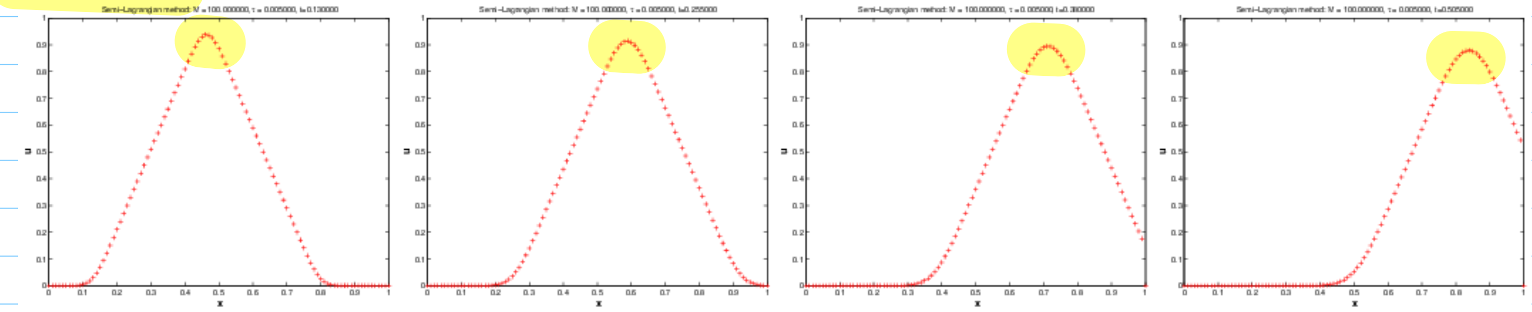
- Linear finite element Galerkin discretization with mass lumping in space
- Semi-Lagrangian method: 1D version of (10.3.4.12)
- Explicit Euler streamline backtracking

We give snapshots of the solution for times $t = 0.124 * j, j = 1,2,3,4$.



· Convection dominates

$\epsilon = 10^{-5}$, $\tau = 0.01$, $h = 0.01$:

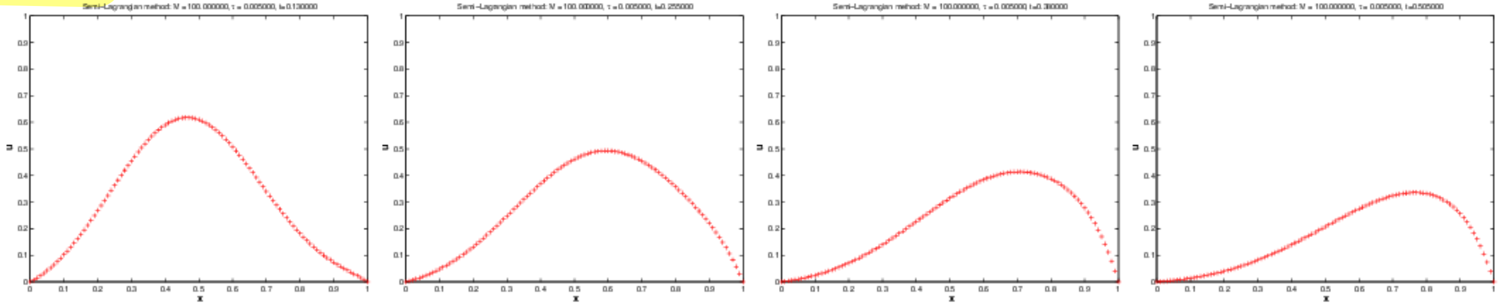


$\rightarrow$ diffusive : rapid smearing of kink (Bad)

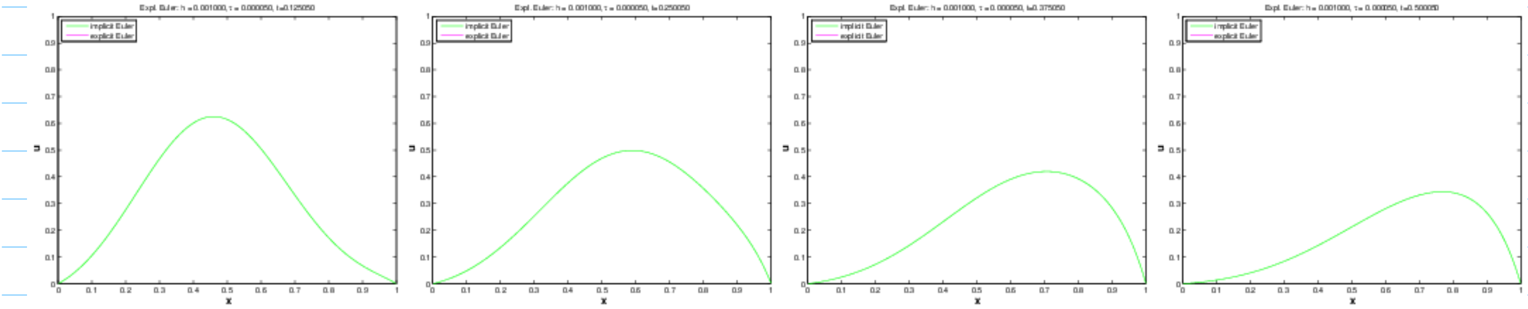
(5)

· Dominant diffusion :

$\epsilon = 0.1$, $\tau = 0.01$, $h = 0.01$:



"Reference solution" computed by method of lines, see Exp. 10.3.1.3, with "overkil resolution" $h = 10^{-3}$, $\tau = 5 \cdot 10^{-5}$:



"reasonable agreement"

▷ Review questions 10.3.4.17