ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 11.1: Conservation Laws: Examples

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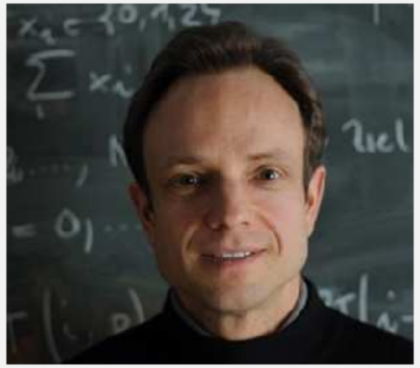


Date: May 15, 2021

Prerequisites.

- Elementary calculus and some fluid dynamics

Dependency. Familiarity with [Lecture → Section 10.1.4] is useful

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

XI. Numerical Methods for Conservation Laws

11.1 Conservation Laws: Examples

11.1.1 Linear Advection

Heat conduction in a fluid, moving with velocity $\mathbf{v}: \Omega \to \mathbb{R}^d$, $d = 1,2,3$; $\Omega \subset \mathbb{R}^d$, $\mathbf{v}_{|\partial\Omega} = 0$

Convective heat flux: $\mathbf{j}(\mathbf{x},t) = \mathbf{v}(\mathbf{x})\,\rho\, u(\mathbf{x},t)$

ρ ≙ heat capacity   temperature

Recall: Conservation law for heat conduction

$$\frac{\partial}{\partial t}(\rho u)(\mathbf{x},t) + (\mathrm{div}_{\mathbf{x}}\,\mathbf{j})(\mathbf{x},t) = f(\mathbf{x},t) \quad \text{in } \tilde{\Omega}. \tag{9.2.1.5}$$

heat source

$$\frac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho \mathbf{v}(\mathbf{x},t) u) = f(\mathbf{x},t) \quad \text{in } \tilde{\Omega} := \Omega \times]0,T[. \tag{10.1.4.1}$$

↳ space-time cylinder

② = A scalar conservation law

Conserved quantity ($f \equiv 0$)

$$\frac{d}{dt}\int_\Omega \rho u \, dx = -\frac{d}{dt}\int_\Omega \operatorname{div}(\rho \mathbf{v} u)\, dx$$

Gauss thm.

$$= -\frac{d}{dt}\int_{\partial\Omega} \rho u \cdot \underbrace{\mathbf{v}\cdot\mathbf{n}}_{=0}\, dS(x) = 0$$

Needed: initial conditions:

$$u(x,0) = u_0(x) \;, x \in \Omega$$

We restrict ourselves to $\Omega = \mathbb{R}^d$

▷ Cauchy problem (→ no spatial boundary conditions)

Simplest case: $d = 1$, $\rho \equiv 1$, $v \equiv \text{const}$, $f \equiv 0$

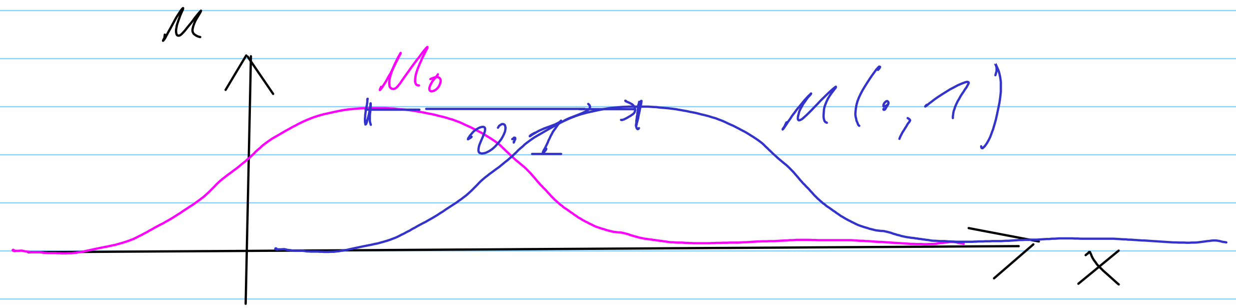
(10.1.4.1) ▷ $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(v u) = 0 \quad \text{on } \mathbb{R}\times[0,T] \qquad (8.1.1.11)$$

$$u(x,0) = u_0(x) \;, x \in \mathbb{R}$$

1D linear advection problem

Solution

$$u(x,t) = u_0(x - vt) \qquad (*)$$



Remark 11.1.1.13: (*) makes perfect sense even for discontinuous u_0!

11.1.2. Traffic Modeling

→ Identical cars on single long highway lane

11.1.2.1 "Particle Model"

$x_i(t) \;\hat{=}\;$ position the i-th car at time t, $i = 1,\dots,N$

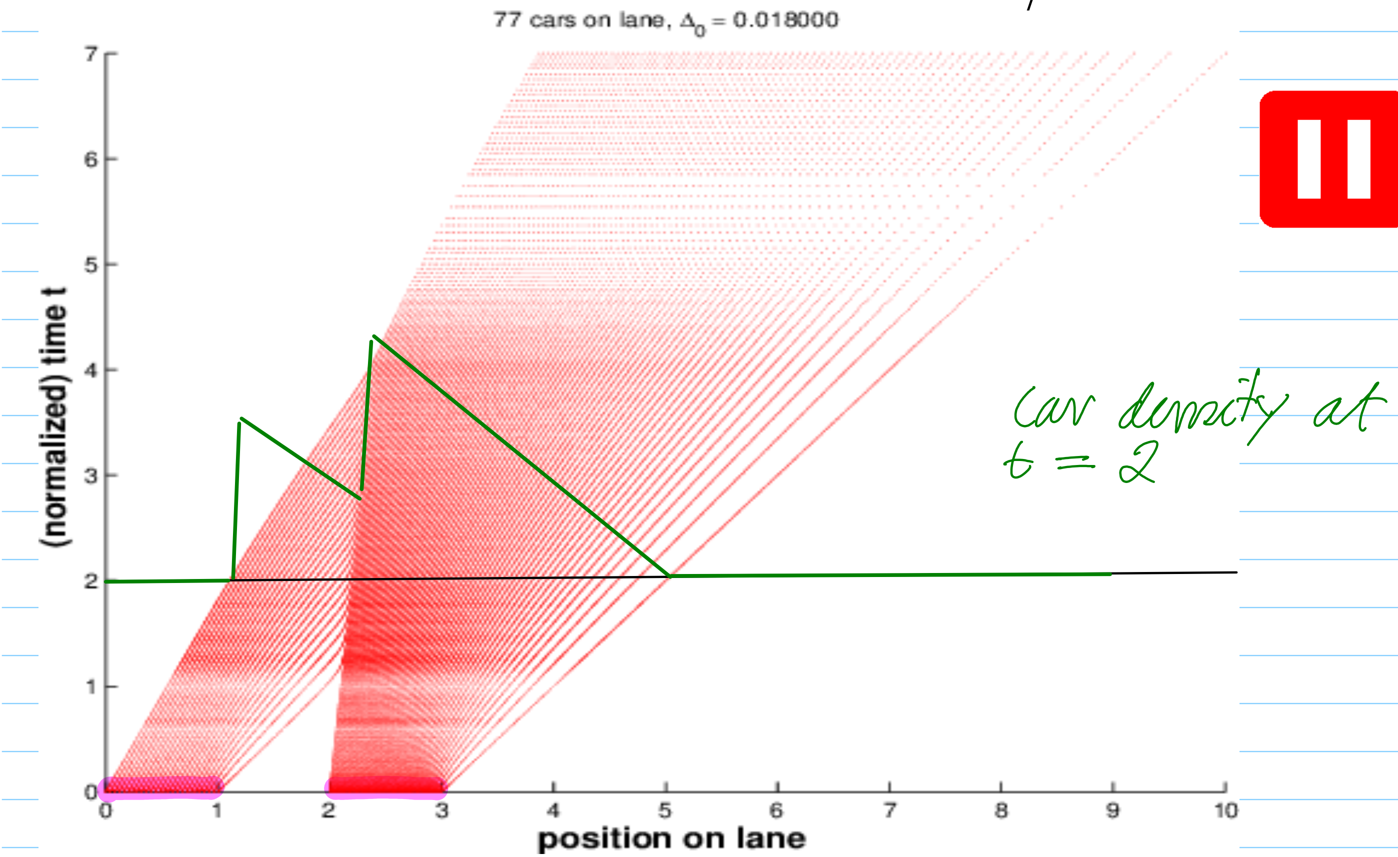
Assume: $x_i(t) < x_{i+1}(t)$

Velocity model: ($\Delta_0 \;\hat{=}\;$ Length of car)

$$\dot{x}_i(t) = v_{max}\left(1 - \frac{\Delta_0}{\Delta x_i(t)}\right), \quad \Delta x_i = x_{i+1} - x_i$$

$$\Delta x_N = \infty$$

③ ▷ ODE in $\mathbb{R}^N$ for car positions



Towards a continuum model :

Limit $N \to \infty$, $\Delta_0 = N^{-1}$

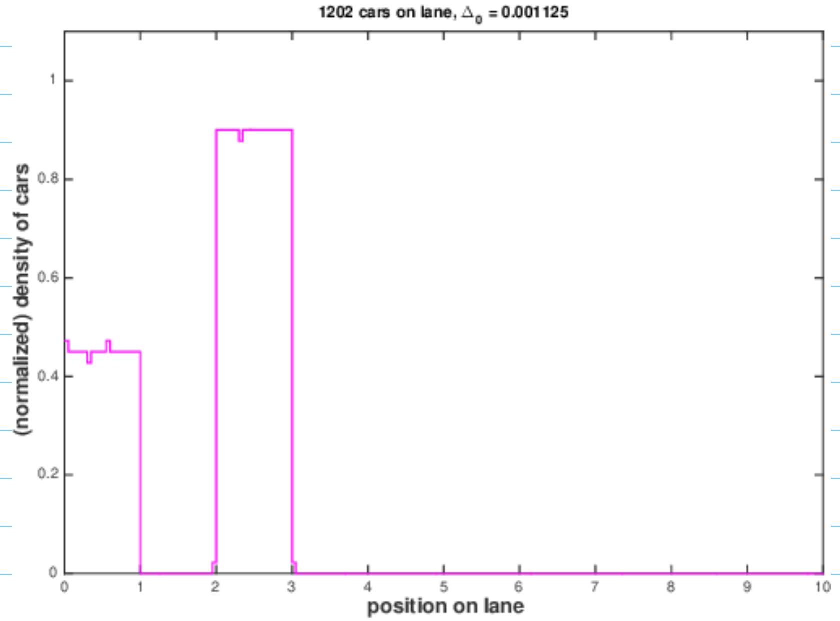
Needed : "Macroscopic quantities" :

Key macroscopic quantity: (normalized) density of cars

$$u_\delta(x,t) := \frac{\Delta_0}{2\delta}\,\sharp\{i \in \{1,\dots,N\} : x-\delta \le x_i(t) < x+\delta\}\,, \qquad \text{(II.1.2.12)}$$

where $\delta > 0$ is the spatial averaging length. ▷ $0 \le u_\delta(x,t) \le 1$

Exp. II.1.2.13: Numerical experiment : $\delta = N^{-1}$

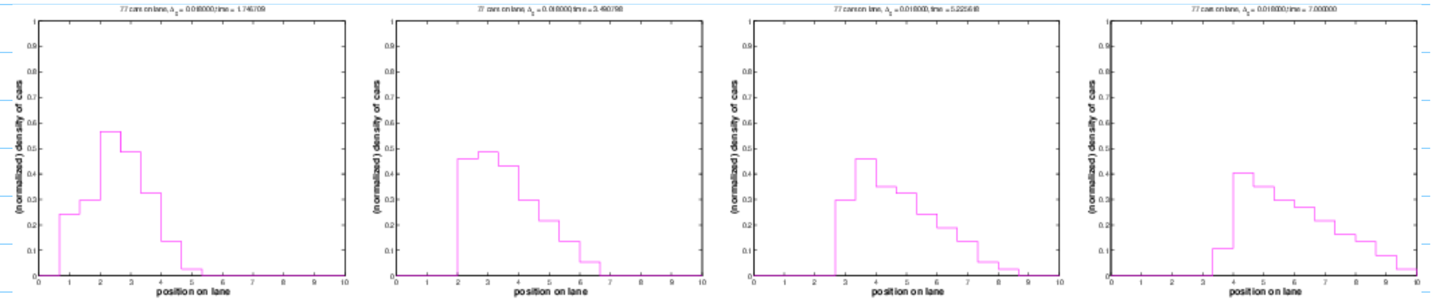


◁ Initial density

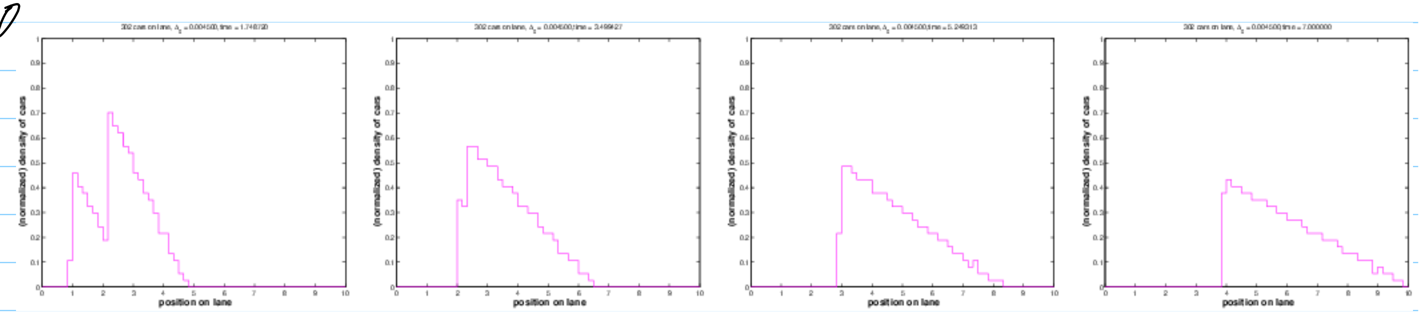
$N = \frac{3}{2} k$

- $k/2$ cars in $[0,1]$
- k cars in $[2,3]$

$k = 50$

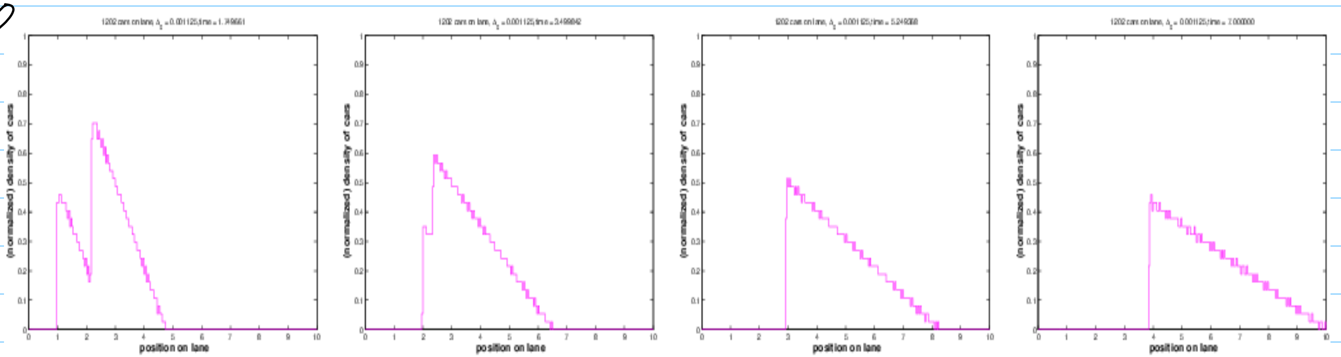


$k = 200$



(4)

$k = 850$



Observed: $u_\delta(x,t) \longrightarrow u(x,t)$

for $N \to \infty \iff \delta \to 0$

II.1.2.2. Continuum Traffic Model

(I): Key macroscopic quantity: (normalized) density of cars

$$u_\delta(x,t) := \frac{\Delta_0}{2\delta}\,\sharp\{i \in \{1,\dots,N\} : x-\delta \le x_i(t) < x+\delta\}\,, \qquad \text{(II.1.2.12)}$$

where $\delta > 0$ is the spatial averaging length.

(II): Required: concept of a macroscopic velocity

Idea: spatial averaging of velocities of cars

$$\boxed{v_\delta(x,t) = \frac{\sum_{i\in\mathcal{U}_\delta(x)} \dot{x}_i(t)}{\sharp\mathcal{U}_\delta}(x)}\,, \qquad \text{(II.1.2.16)}$$

$$\mathcal{U}_\delta(x) := \{i \in \{1,\dots,N\} : x-\delta \le x_i(t) < x+\delta\}\,.$$

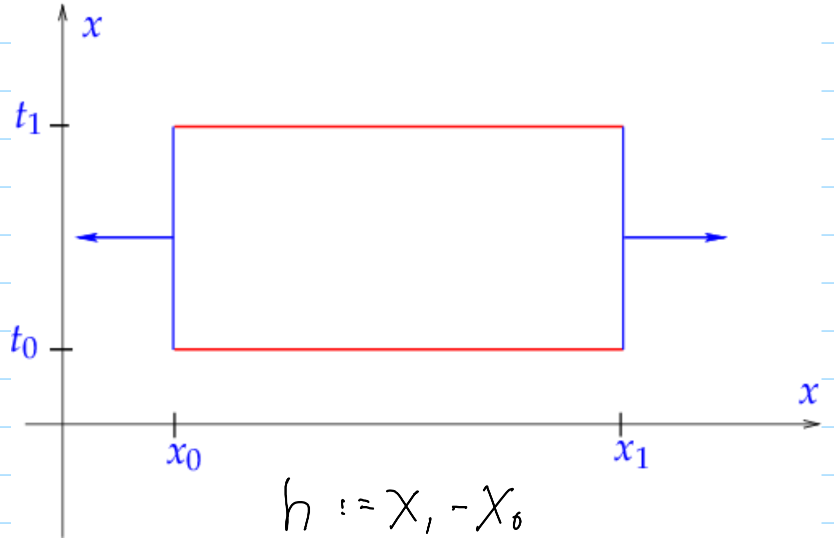
no of cars in $[x-\delta, x+\delta]$

No. of cars passing the site x in unit time

Flux of cars: $q_\delta(x,t) = u_\delta(x,t)\,v_\delta(x,t)$

▷ Conservation of cars:

$\tau := t_1 - t_0$



$h := x_1 - x_0$

$$\underbrace{\int_{x_0}^{x_1} u_\delta(x,t_1)\,\mathrm{d}t - \int_{x_1}^{x_1} u_\delta(x,t_0)\,\mathrm{d}t}_{\text{change of no. of cars on } [x_0,x_1] \text{ in } [t_0,t_1]} \approx \underbrace{\int_{t_0}^{t_1} q_\delta(x_0,t)\,\mathrm{d}x - \int_{t_0}^{t_1} q_\delta(x_1,t)\,\mathrm{d}x}_{\text{no. of cars entering/leaving } [x_0,x_1] \text{ in } [t_0,t_1]}\,. \qquad \text{(II.1.2.18)}$$

limit $N \to \infty$, $\Delta_0 := N^{-1} \to 0$, $u_\delta \to u$, $q_\delta \to q$

▷ Limit balance law

$$\underbrace{\int_{x_0}^{x_1} u(x,t_1)\,\mathrm{d}t - \int_{x_1}^{x_1} u(x,t_0)\,\mathrm{d}t}_{\text{change of no. of cars on } [x_0,x_1] \text{ in } [t_0,t_1]} = \underbrace{\int_{t_0}^{t_1} q(x_0,t)\,\mathrm{d}x - \int_{t_0}^{t_1} q(x_1,t)\,\mathrm{d}x}_{\text{no. of cars entering/leaving } [x_0,x_1] \text{ in } [t_0,t_1]}\,. \qquad \text{(II.1.2.20)}$$

5 Differential form of balance law

Assume : u, q smooth $\triangleright$ Taylor expansion

$$\int_{x_0}^{x_1} u(x,t_1) - u(x,t_0)\,\mathrm{d}x = h(u(x_0,t_1) - u(x_0,t_0)) + O(h^2) \quad \text{for} \quad h \to 0\,,$$

$$\int_{t_0}^{t_1} q(x_1,t) - q(x_0,t)\,\mathrm{d}t = \tau(q(x_1,t_0) - q(x_0,t_0)) + O(\tau^2) \quad \text{for} \quad \tau \to 0\,.$$

$$u(x_0,t_1) - u(x_0,t_0) = \frac{\partial u}{\partial t}(x_0,t_0)\tau + O(\tau^2) \quad \text{for} \quad \tau \to 0\,,$$

$$q(x_1,t_0) - q(x_0,t_0) = \frac{\partial q}{\partial x}(x_0,t_0)h + O(h^2) \quad \text{for} \quad h \to 0\,.$$

Balance law $\triangleright$ $\cancel{h}\tau \dfrac{\partial u}{\partial t}(x_0,t_0) = \cancel{h}\tau \dfrac{\partial q}{\partial x}(x_0,t_0) + O(\tau^2+h^2)$

For $h,\tau \to 0$

$$\blacktriangleright \quad \frac{\partial u}{\partial t}(x,t) + \frac{\partial q}{\partial x}(x,t) = 0 \quad \text{in } \Omega\times]0,T[\,. \qquad \text{(II.1.2.21)}$$

Next : constitutive relationship $q = q(u)$

Average : $\dot{x}_i = 1 - \dfrac{\Delta_0}{\Delta x_i}$

$\downarrow \qquad\qquad \downarrow$

$v_\delta(x,t) = 1 - u_\delta(x,t)$

$N \to \infty$:

$$v(x,t) = 1 - u(x,t) \overset{(8.1.2.17)}{\Rightarrow} q(x,t) = u(x,t)(1-u(x,t))\,. \qquad \text{(II.1.2.22)}$$

(II.1.2.21) & (II.1.2.22) & (II.1.2.17) $\blacktriangleright$ $\boxed{\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(u(1-u)) = 0 \quad \text{in } \Omega\times]0,T[}$. (II.1.2.23)

+ macroscopic counterpart of initial conditions (8.1.2.7):

$$u(x,0) = u_0(x)\,, \quad x \in \mathbb{R}\,. \qquad \text{(II.1.2.24)}$$

A scalar conservation law

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Review questions 11.1.3.7

[Please try to answer without looking at (lecture) notes]

A:

Consider the Cauchy problem

$$\frac{\partial u}{\partial t} + \operatorname{div}(\mathbf{v}(\boldsymbol{x})u) = 0 \quad \text{in} \quad \widetilde{\Omega} := \mathbb{R}^d \times]0,T[\,, \qquad (8.1.1.5)$$
$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^d \quad \text{(initial conditions)}\,.$$

for linear advection for $d = 2$ and the velocity field $\mathbf{v}(\boldsymbol{x}) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$. Write down the solution $u = u(\boldsymbol{x},t)$ in terms of the initial data $u_0 = u_0(\boldsymbol{x})$.

B:

Let $\boldsymbol{\Phi} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be the flow induced by a smooth velocity field $\mathbf{v} : \mathbb{R}^d \to \mathbb{R}^d$. Which partial differential equations does the function

$$u(\boldsymbol{x},t) := u_0(\widehat{\boldsymbol{x}})\,, \quad \boldsymbol{x} = \boldsymbol{\Phi}(\widehat{\boldsymbol{x}},t)\,, \quad \widehat{\boldsymbol{x}} \in \mathbb{R}^d$$

satisfy. Here $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a given differentiable function with bounded support.

C:

In an $x - t$ diagram sketch the trajectory of a car starting at $t = 0$, $x = 0$ and moving with *constant acceleration* to right.

D:

Which traffic flow conservation law arises, when the speed law

$$v_{\text{opt}}(\Delta x) = v_{\max}\left(1 - \frac{\Delta_0}{\Delta x}\right)$$

is replaced with

$$v_{\text{opt}}(\Delta x) = v_{\max} \cos\left(\frac{\pi}{2}\frac{\Delta_0}{\Delta x}\right)\,.$$