ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 11.2.1: Scalar Conservation Laws in 1D: Integral and Differential Form

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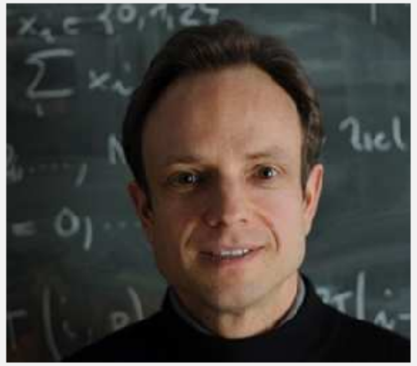
Date: May 15, 2021



Prerequisites.

- Calculus in 1D and higher dimensions

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 $\to$ New Chapter 9 , Old Chapter 8 $\to$ New Chapter 11
Trailing digits in labels are not affected.

XI. Numerical Methods for Conservation Laws

11.2 Scalar Conservation Laws [in 1D]

11.2.1 Integral and Differential Form

Space-time cylinder $\tilde{\Omega} = \Omega \times [0,T]$, $\Omega \subset \mathbb{R}^d$

Integral form of scalar conservation law:

> Conservation law for transient state distribution $u : \tilde{\Omega} \mapsto \mathbb{R}$: $u = u(\boldsymbol{x},t)$, for $0 \le t \le T$:
>
> $$\underbrace{\frac{d}{dt}\int_V u \,\mathrm{d}\boldsymbol{x}}_{\text{change of amount in } V} + \underbrace{\int_{\partial V} \mathbf{f}(u,\boldsymbol{x})\cdot\mathbf{n}\,\mathrm{d}S(\boldsymbol{x})}_{\text{inflow/outflow}} = \underbrace{\int_V s(u,\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}}_{\text{production term}} \quad \forall \text{ "control volumes" } V \subset \Omega . \qquad (11.2.1.1)$$

↳ flux function : $f : \mathbb{R}\times\Omega \to \mathbb{R}^d$

⇔ Balance for transient heat conduction, with $u \triangleq$ temperature distribution

Fourier law : " $f(u,x) = -\kappa(x)\,\mathrm{grad}\, u$ "
[diffusive flux]

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Assumptions on flux

- depends only on x and the local state $u(x,t)$
- may be non-linearly depend on u

Recast (II.2.1.1)

(I) Integration in time

▶ Space-time integral form of (8.2.1.1), cf. (8.1.3.2),

$$\int_V u(x,t_1)\,\mathrm{d}x - \int_V u(x,t_0)\,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\partial V} \mathbf{f}(u,x)\cdot n\,\mathrm{d}S(x)\mathrm{d}t = \int_{t_0}^{t_1}\int_V s(u,x,t)\,\mathrm{d}x\mathrm{d}t \qquad \text{(II.2.1.4)}$$

for all $V\subset\Omega$, $0<t_0<t_1<T$, $n \mathrel{\hat=}$ exterior unit normal at ∂V

(II) Apply Gauss theorem $\int_{\partial V} f\cdot n\, dS = \int_V \operatorname{div} f\, dx$

▶ (local) differential form of (8.2.1.1):

$$\boxed{\frac{\partial}{\partial t}u + \operatorname{div}_x \mathbf{f}(u,x) = s(u,x,t) \quad \text{in } \tilde{\Omega}} \,. \qquad \text{(II.2.1.5)}$$

div acting on spatial variable x only

Differential form for $d=1$:

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}(f(u(x,t),x)) = s(u(x,t),x,t) \quad \text{in }]\alpha,\beta[\times]0,T[,\quad \alpha,\beta\in\mathbb{R}\cup\{\pm\infty\}\,. \qquad \text{(II.2.1.6)}$$

↳ $f: \mathbb{R}\times\Omega \to \mathbb{R}$, $\Omega\subset\mathbb{R}^d$

If $\Omega=\mathbb{R}$ ⟹ Cauchy problem

Complete statement only requires initial condition

$u(x,0) = u_0(x)$, $x\in\mathbb{R}$

Examples:

linear advection: $\frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\mathbf{v}(x,t)(\rho u)) = f(x,t)$ in $\Omega\times]0,T[$, $(\Omega\subset\mathbb{R}^d)$, (II.1.1.3)

Burgers equation: $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$ in $\Omega\times]0,T[$, $(\Omega\subset\mathbb{R})$, (II.1.3.4)

traffic flow equation: $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u(1-u)) = 0$ in $\Omega\times]0,T[$, $(\Omega\subset\mathbb{R})$, (II.1.2.23)

II Identify flux functions (& source function)

- ✦ For Burgers equation (II.1.3.4): $f(u,x) = f(u) = \frac{1}{2}u^2$, $s=0$,
- ✦ For traffic flow equation (II.1.2.23): $f(u,x)=f(u)=u(1-u)$, $s=0$,
- ✦ For linear advection (II.1.1.3): $\mathbf{f}(u,x) = \mathbf{v}(x,t)u$, $s=f(x,t)$

(Note: in this case the conserved quantity is actually ρu, which was again denoted by u)

③

For linear advection the flux is a linear function of u.

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Review questions 11.2.1.9

[Answer without aids]

A:

The differential form of a generic scalar conservation law is

$$\frac{\partial}{\partial t}u + \operatorname{div}_{\boldsymbol{x}}\mathbf{f}(u,\boldsymbol{x}) = s(u,\boldsymbol{x},t) \quad \text{in} \quad \widetilde{\Omega} := \Omega \times [0,T] \,, \tag{8.2.1.5}$$

with flux function $\mathbf{f} : U \times \Omega \to \mathbb{R}^d$ and source function s. What is the integral form of (8.2.1.5)?

B:

Identify the flux functions and source functions for the following scalar conservation laws:

linear advection: $\frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\mathbf{v}(\boldsymbol{x},t)(\rho u)) = f(\boldsymbol{x},t) \quad \text{in} \quad \Omega\times]0,T[\,, \quad (\Omega \subset \mathbb{R}^d)\,,$ (8.1.1.3)

Burgers equation: $\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0 \quad \text{in } \Omega\times]0,T[\,, \quad (\Omega \subset \mathbb{R})\,,$ (8.1.3.4)

traffic flow equation: $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u(1-u)) = 0 \quad \text{in } \Omega\times]0,T[\,, \quad (\Omega \subset \mathbb{R})\,.$ (8.1.2.23)

C:

The heat equation $\frac{\partial u}{\partial t} - \Delta u = 0$ also fits (8.2.1.5) when the flux function is chosen appropriately. What is this so-called diffusive flux?