ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 11.2.2: Scalar Conservation Laws in 1D: Characteristics

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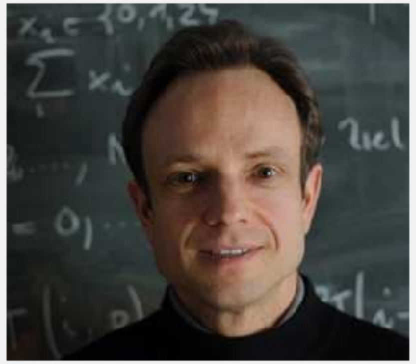


Dependency. [Lecture → Section 11.2.1]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

XI. Numerical Methods for Conservation Laws

11.2 Scalar Conservation Laws [in 1D]

11.2.2 Characteristics

Cauchy problem for 1D scalar conservation law

(CL) (11.2.2.1)

$$\partial_t u + \partial_x f(u) = 0 \quad \text{on } \mathbb{R}\times[0,T]$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R}$$

with C^1 flux function $f:\mathbb{R}\to\mathbb{R}$

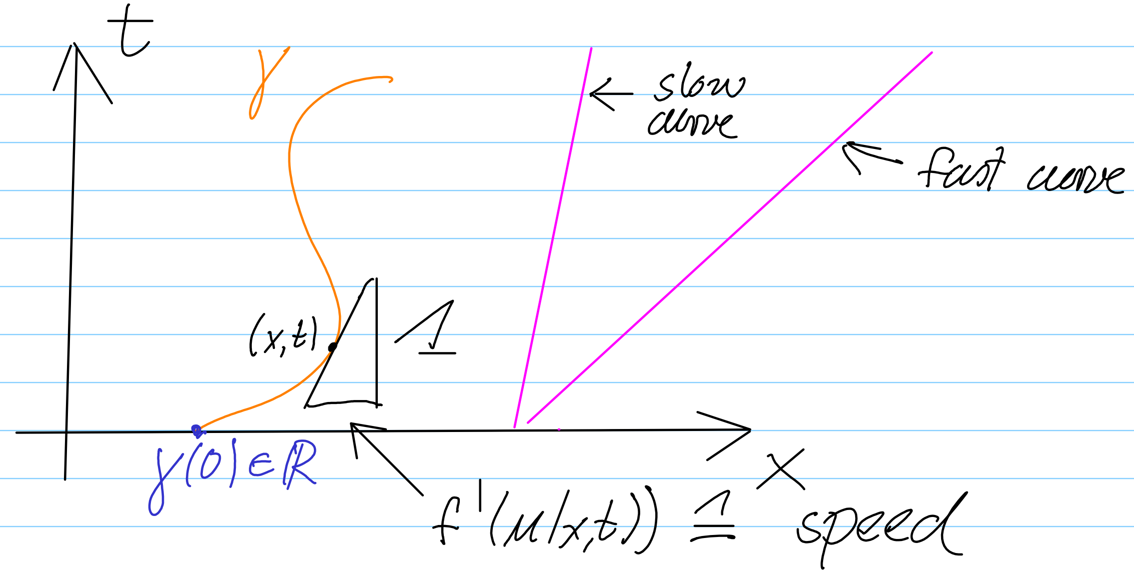
$u \hat{=}$ smooth solution of (CL)

Def.: (11.2.2.3) Characteristic (curve) $\gamma:[0,T]\to\mathbb{R}$ for (CL)

$$\dot{\gamma}(\tau) = f'(u(\gamma(\tau),\tau)), \quad f' = \frac{df}{du}$$

→ family of curves

②



This is a more general fact, because by chain rule

$$\frac{d}{dt} u(\gamma(t),t) = \frac{\partial u}{\partial x}(\gamma(t),t)\,\dot{\gamma}(t) + \frac{\partial u}{\partial t}(\gamma(t),t)$$

$$= \frac{\partial u}{\partial x}(\gamma(t),t)\, f'(u(\gamma(t),t)) + \frac{\partial u}{\partial t}(\gamma(t),t)$$

$$= \frac{\partial u}{\partial t}(\gamma(t),t) + \frac{\partial}{\partial x} f(u)\Big|_{(x,t)=(\gamma(t),t)}$$

$$= 0$$

provided that u is C^1-smooth.

Ex II.2.2.5: Linear advection with constant velocity

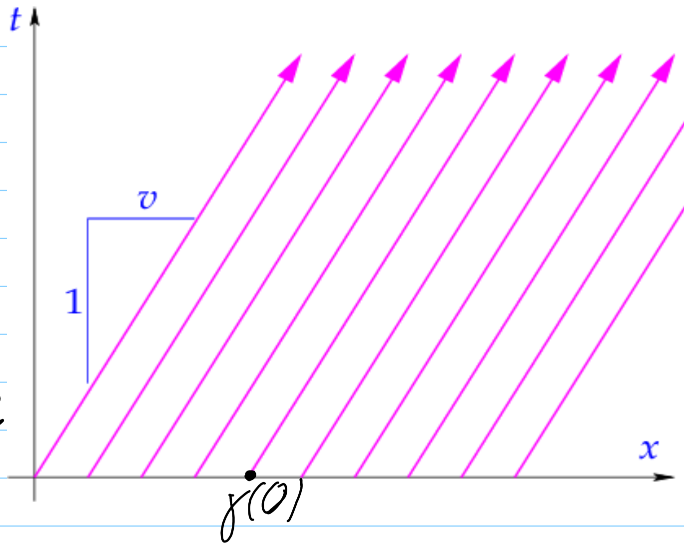
$$f(u) = vu, \quad v \in \mathbb{R}$$

$f'(u) = v$

$\dot{\gamma}(\tau) = v$

$\gamma(\tau) = v\tau + x_0$, $x_0 \in \mathbb{R}$

$\gamma = \{(x,t) : x - vt = x_0\}$



General solution for C.P.: $u(x,t) = u_0(x - vt)$

▷ $u_{|\gamma} = \text{const}$ $\forall$ characteristics γ

Lemma II.2.2.6. Classical solutions and characteristic curves

Smooth solutions of (II.2.2.1) are constant along characteristic curves.
(CL)

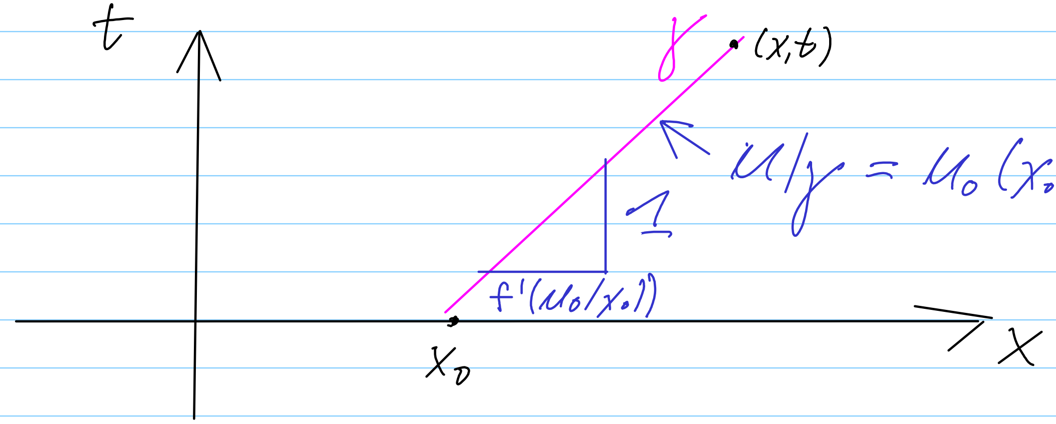
$u_{|\gamma} \equiv \text{const}$, if γ characteristic
$[\ \dot{\gamma}(\tau) = f'(u(\gamma(t),t))\ ]$

▷ γ has constant slope

▷ Characteristics are straight lines in the x-t plane

$$\gamma = \{(x_0 + t f'(u_0(x_0)), t),\ 0 \le t \le T\}$$

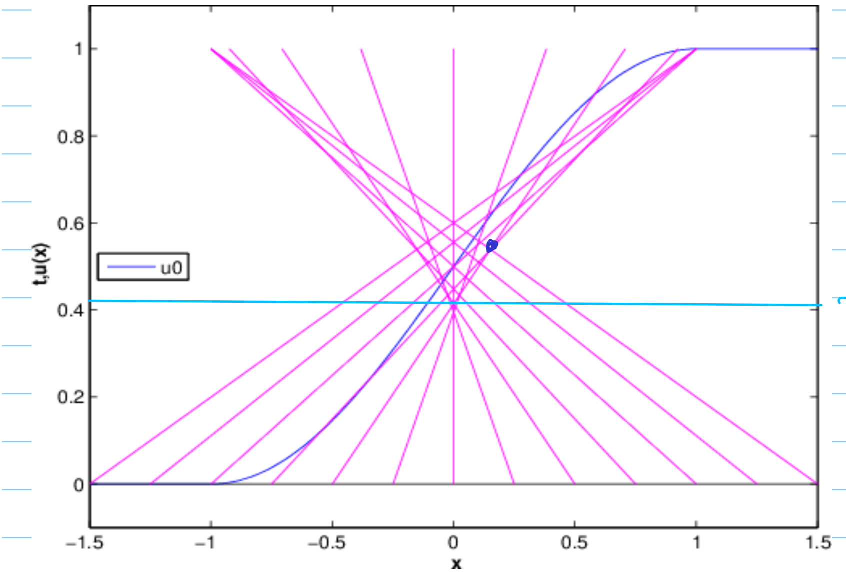
▷ Geometric construction



▷ Implicit solution formula

$$u(x,t) = u_0\big(x - f'(u(x,t))\,t\big) \qquad (11.2.2.7)$$

§11.2.2.8: Example: Traffic flow, $f(u) = u(1-u)$

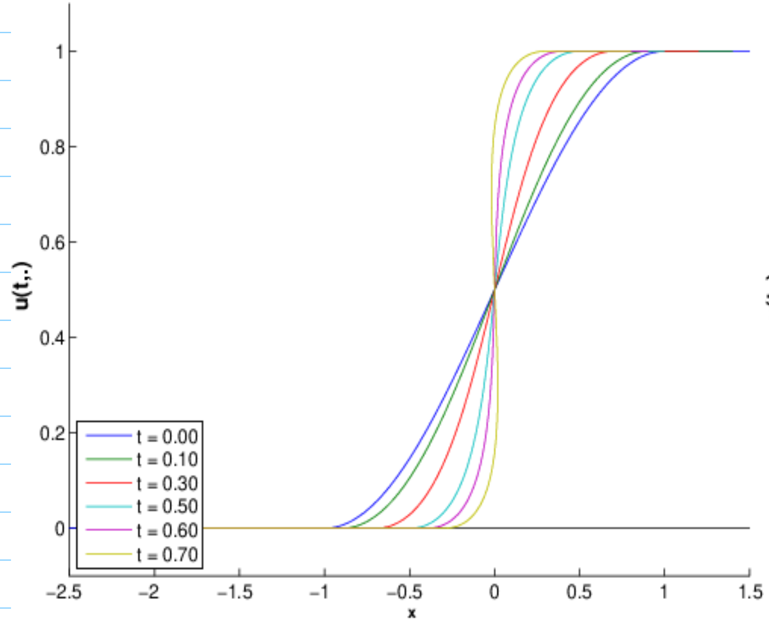


$f'(u) = 1 - 2u$

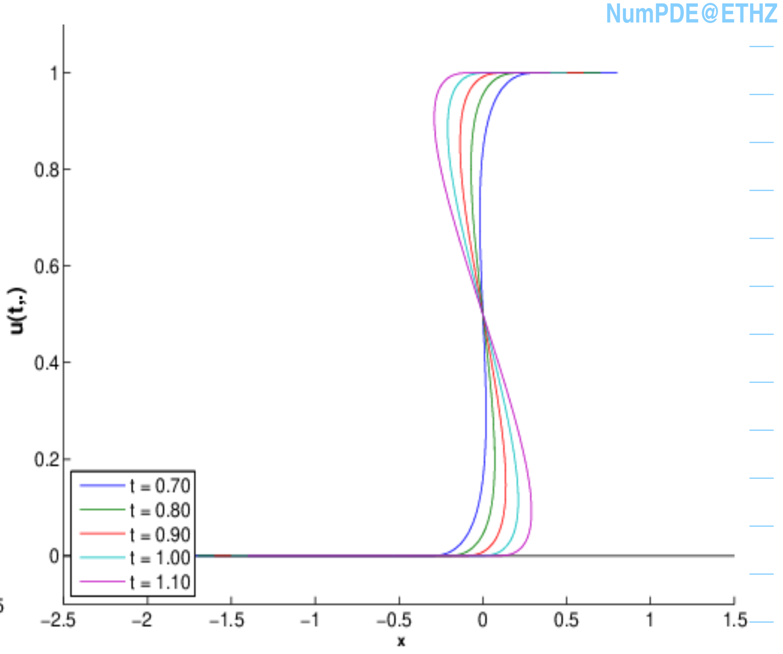
◁ Characteristics intersect

$t \approx 0.7$

solution formula fails in case of intersecting characteristics

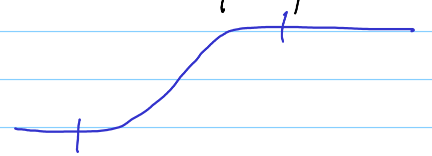


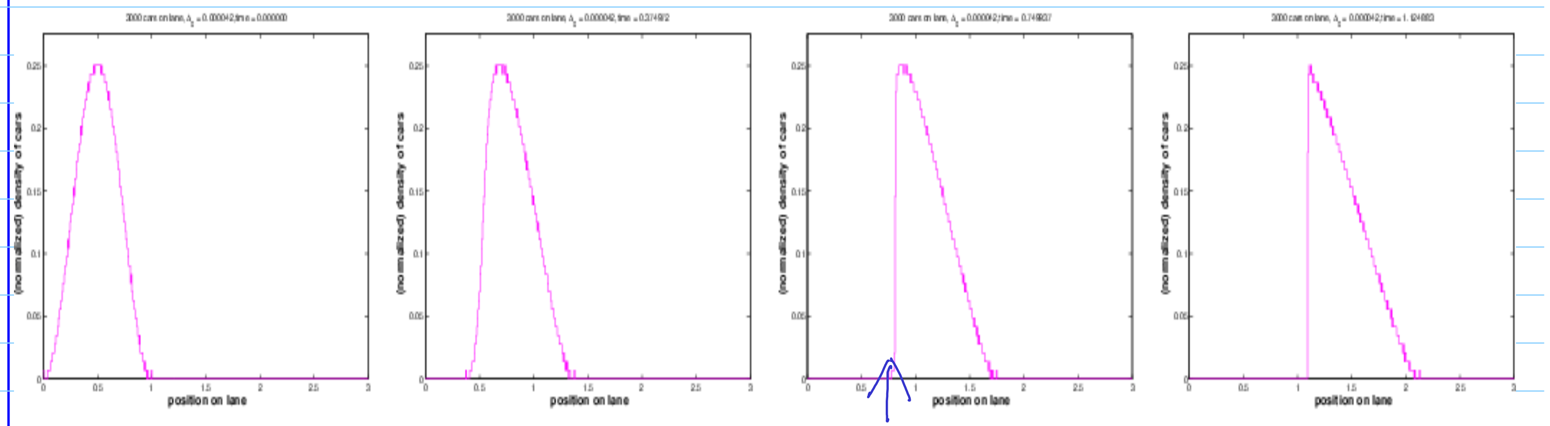
$t < 0.7$: solution by (8.2.2.7)



the wave breaks: "multivalued solution"

Ex 11.2.2.10: Traffic, solution of particle model, $N=5000$

$u_0(x) =$ 



discontinuity emerges

Meaning of characteristic curves :

$\hat{=}$ Information propagates along characteristics

5 Review questions 11.2.2.11

[Answer without resorting to lecture/tablet notes]

A:

In the $x-t$-plane sketch the characteristics for the Cauchy problem for the scalar conservation law

$$\begin{aligned} \frac{\partial u}{\partial t}+\frac{\partial}{\partial x}f(u) &= 0 \quad \text{in } \mathbb{R}\times]0,T[\,, \\ u(x,0) &= u_0(x) \quad \text{in } \mathbb{R}\,. \end{aligned} \tag{8.2.2.1}$$

with flux function $f(u)=u^2$ and initial data

$$u_0(x)=\begin{cases} 1+x & \text{for } -1<x\leq 0\,, \\ 1-x & \text{for } 0<x\leq 1\,, \\ 0 & \text{elsewhere.} \end{cases}$$

B:

For a scalar 1D conservation law with flux functions

1. $f(u)=u^2$,
2. $f(u)=\sin(\pi u)$,
3. $f(u)=\cos(\pi u)$

and initial data $u_0(x)=1$ for $-1\leq x\leq 1$, $u_0(x)=0$ elsewhere, sketch the family of characteristic curves in an $x-t$ diagram.

C:

Directly verify that, if well-defined, the implicitly defined function

$$u(x,t)=u_0(x-f'(u(x,t))t)\,,\quad (x,t)\in\mathbb{R}\times[0,T]\,, \tag{8.2.2.7}$$

with a strictly convex and smooth flux function $f:\mathbb{R}\to\mathbb{R}$ provides a solution of (8.2.2.1).

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