ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 11.3.4: Numerical Flux Functions

Prof. R. Hiptmair, SAM, ETH Zurich

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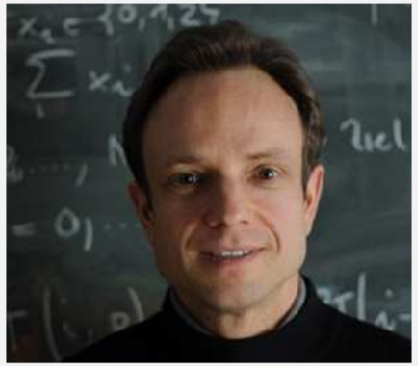


Dependency. [Lecture → Section 11.2.1] and [Lecture → Section 11.3.2]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

XI Numerical Methods for Conservation Laws

11.3. Conservative Finite-Volume (FV) Discretization

11.3.4. Numerical Flux Functions

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad \text{in } \mathbb{R}\times]0,T[\quad , \quad u(x,0) = u_0(x) \; , \quad x \in \mathbb{R} \; . \qquad (11.2.2.1)$$

and its conservative finite volume discretization on an (infinite) equidistant spatial mesh with mesh width $h > 0$:

$$\frac{d\mu_j}{dt}(t) = -\frac{1}{h}\left(F(\mu_{j-m_l+1}(t),\dots,\mu_{j+m_r}(t)) - F(\mu_{j-m_l}(t),\dots,\mu_{j+m_r-1}(t))\right) \; , \quad j \in \mathbb{Z} \; . \qquad (11.3.2.7)$$

We abbreviate $f_{j+\frac{1}{2}}(t) := F(\mu_{j-m_l+1}(t),\dots,\mu_{j+m_r}(t))$.

Special case: 2-point numerical flux ($m_l = m_r = 1$): $F = F(v,w)$

($v \triangleq$ left state, $w \triangleq$ right state)

(8.3.2.7) ▶ $$\frac{d\mu_j}{dt}(t) = -\frac{1}{h}\left(F(\mu_j(t),\mu_{j+1}(t)) - F(\mu_{j-1}(t),\mu_j(t))\right) \; , \quad j \in \mathbb{Z} \; . \qquad (11.3.2.8)$$

Goal $F(\mu_j(t), \mu_{j+1}(t)) \approx f(u(x_{j+1/2}, t))$

$\mu_j(t) \approx u(t, x_j)$

Definition II.3.3.5. Consistent numerical flux function

A numerical flux function $F : \mathbb{R}^{m_l+m_r} \mapsto \mathbb{R}$ is consistent with the flux function $f : \mathbb{R} \mapsto \mathbb{R}$, if

$$F(u, \dots, u) = f(u) \quad \forall u \in \mathbb{R}.$$

II.3.4.1 Central flux

$$F_c(v, w) = \begin{cases} \tfrac{1}{2}(f(v) + f(w)), & (1) \\ f(\tfrac{1}{2}(v+w)) & (2) \end{cases}$$

↳ consistent

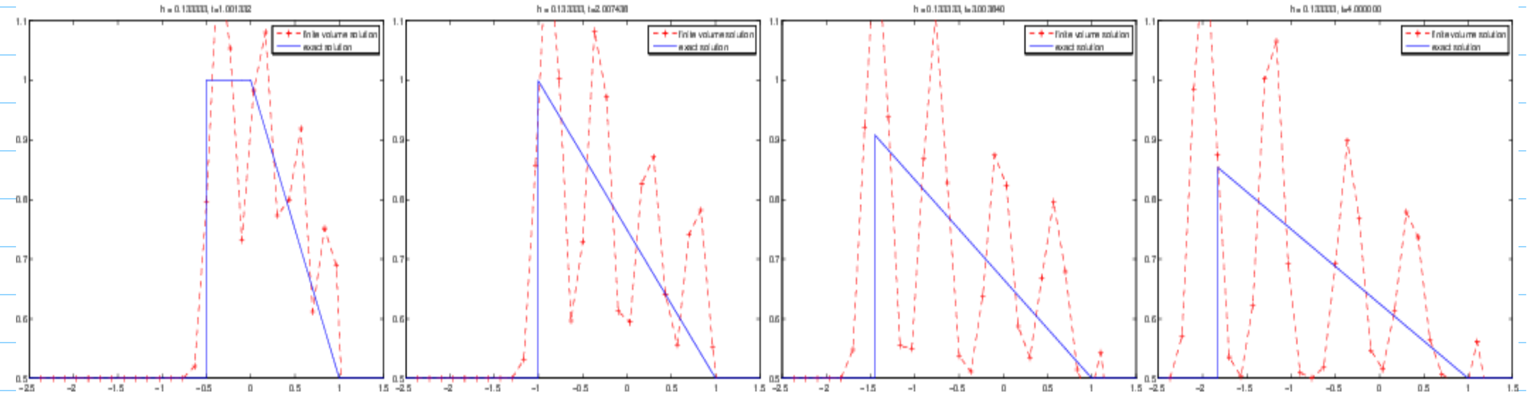
▷ FV-MOL-ODE:

(1): $\dot{\mu}_j = -\frac{1}{2h}\left(f(\mu_{j+1}) - f(\mu_{j-1})\right)$

(2): $\dot{\mu}_j = -\frac{1}{h}\left(f(\tfrac{1}{2}(\mu_j + \mu_{j+1})) - f(\tfrac{1}{2}(\mu_{j-1} + \mu_j))\right)$

Exp: F_c for traffic flow C.P.

$u_0 = \chi_{[0,1]}$



▷ μ_N useless: Strong spurious oscillations

II.3.4.2 Lax-Friedrichs / Rusanov Flux

Idea: Suppress oscillations by adding artificial diffusion

diffusive flux

Ex: linear advection, velocity $c > 0$

$$\partial_t u + \underbrace{c\,\partial_x u - \varepsilon\,\partial_x^2 u}_{= \partial_x(\underbrace{c u - \varepsilon \partial_x u}) } = 0$$

Flux function: $f(u) = c u - \varepsilon \partial_x u$

(3) $f(u) = cu - \kappa\,\partial_x u$

$$F(v,w) = \underbrace{c\,\tfrac{1}{2}(v+w)}_{\text{central flux}} - \underbrace{\kappa\,\frac{w-v}{h}}_{\text{difference quotient}}$$

II $$\dot{\mu}_j = -\frac{c}{2h}(\mu_{j+1}-\mu_{j-1}) + \kappa\,\frac{\mu_{j+1}-2\mu_j+\mu_{j-1}}{h^2}$$

Required: $\kappa \longrightarrow 0$ as $h \longrightarrow 0$

$\kappa = O(h)$: $\kappa = \tfrac{1}{2} h c$

(h-dependent diffusion)

For general C.L.:

$$\partial_t u + \partial_x f(u) = \partial_t u + \underbrace{f'(u)}_{\text{(local) advection velocity}}\partial_x u = 0$$

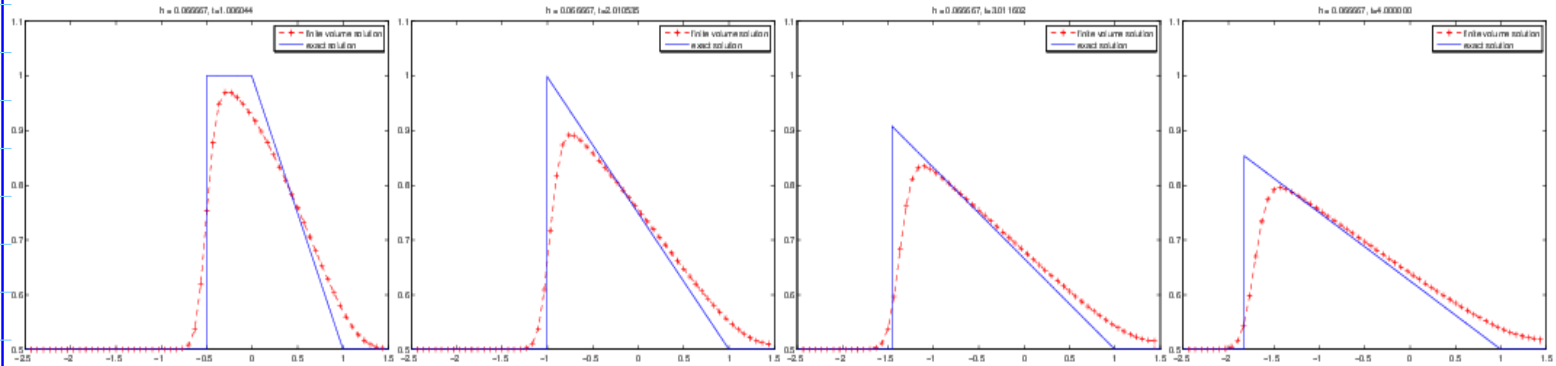
▷ "$\kappa = \tfrac{1}{2} h f'(u)$"

▶ (local) Lax-Friedrichs/Rusanov flux

$$\boxed{F_{\mathrm{LF}}(v,w) = \tfrac{1}{2}(f(v)+f(w)) - \tfrac{1}{2}(w-v)\cdot \max_{\min\{v,w\}\le u\le \max\{v,w\}} |f'(u)|}\;. \qquad \text{(II.3.4.16)}$$

max → add sufficiently strong artificial diffusion

Exp: Traffic flow, $u_0 = \chi_{[0,1]}$, $h = 0.03$



▷ - no more spurious solutions

- smearing of shock

II.3.4.3 Upwind flux

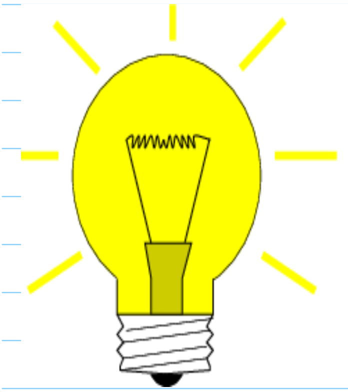
Challenge: $F(\mu_j,\mu_{j+1}) \approx f(u(x_{j+1/2},t))$

Idea: Follow the local direction of propagation of information.

$\Updownarrow$

characteristics, slope $f'(u(x_{j+1/2},t))$

$\uparrow$ unknown



Idea: There is a "velocity of propagation" even at discontinuities of u! Deduce it from Rankine-Hugoniot jump condition (II.2.4.2).

▶ local velocity of transport $= \begin{cases} f'(u) & \text{for unique state, } u = u_l = u_r \\ \dfrac{f(u_r)-f(u_l)}{u_r-u_l} & \text{at discontinuity.} \end{cases}$

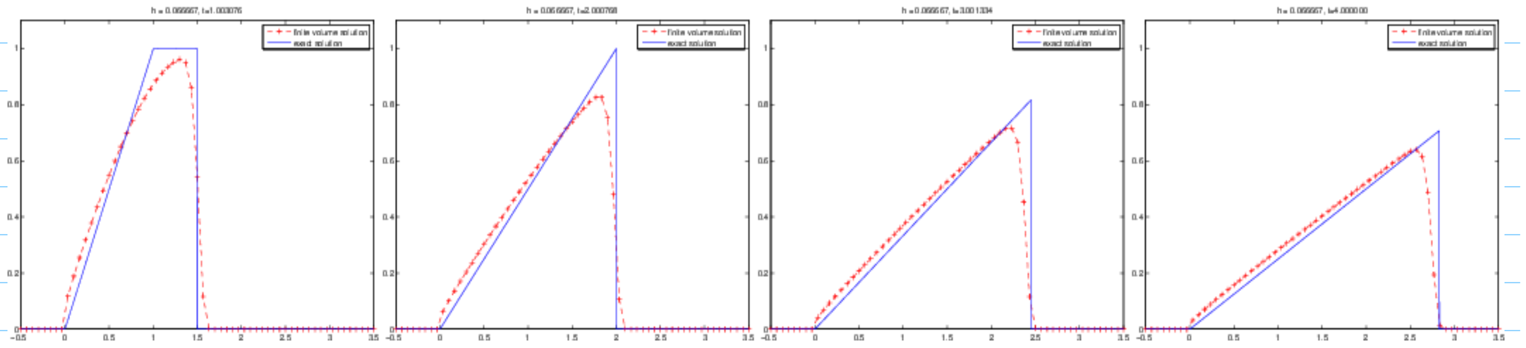
(u_l, u_r $\hat{=}$ states to left and right of discontinuity)

▶ upwind numerical flux for scalar conservation law with flux function f:

$$\boxed{F_{\mathrm{uw}}(v,w) = \begin{cases} f(v) & \text{, if } \dot{s} \geq 0\,, \\ f(w) & \text{, if } \dot{s} < 0\,, \end{cases} \qquad \dot{s} := \begin{cases} \dfrac{f(w)-f(v)}{w-v} & \text{for } v \neq w\,, \\ f'(v) & \text{for } v = w\,. \end{cases}} \tag{II.3.4.19}$$

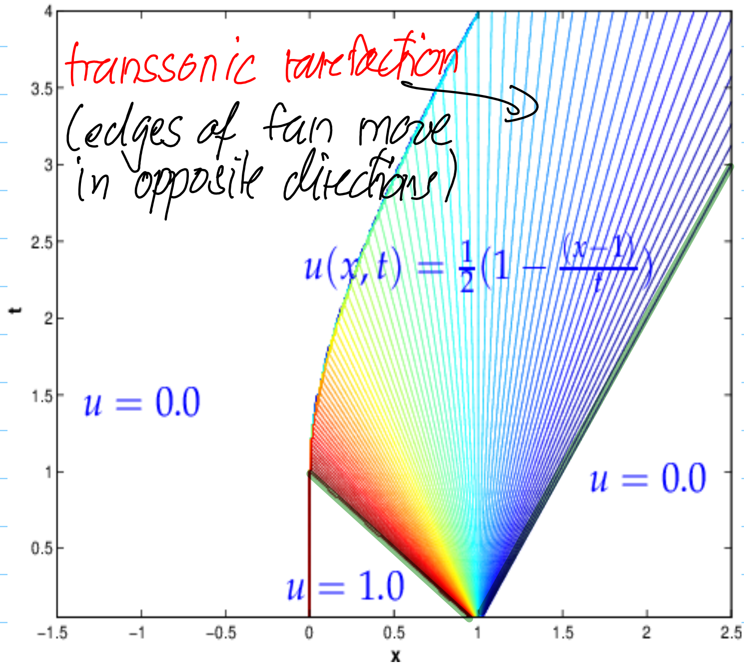
Exp. II.3.4.21: F_{uw} for traffic flow

$$u_0(x) = \begin{cases} 1\,, & 0 \leq x \leq 1 \\ 1/2\,, & \text{elsewhere} \end{cases}$$



- less diffusive than F_{LF}

Exp.: $u_0 = \chi_{[0,1]}$



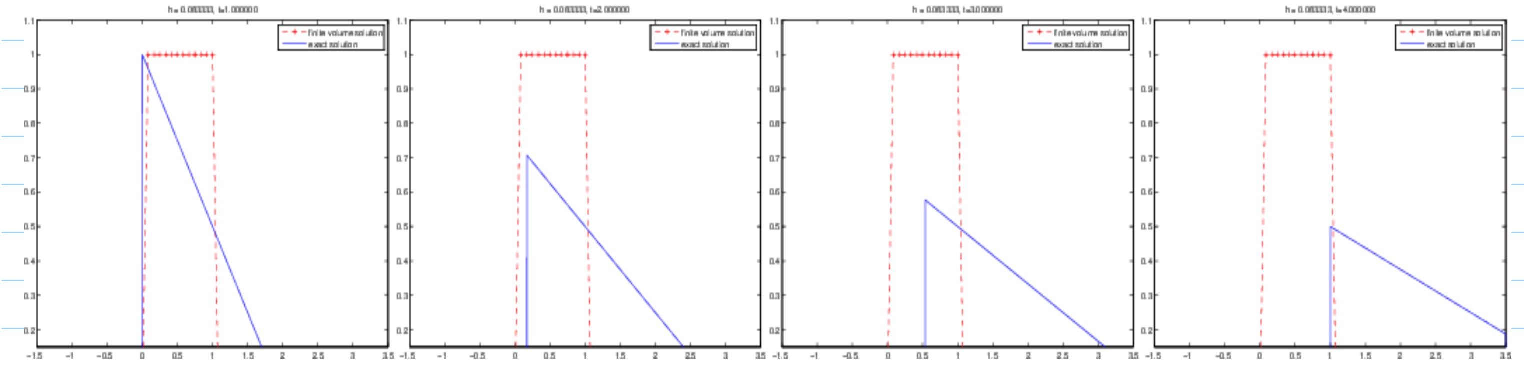
◁ Structure of solution

$u_0(x) \in \{0,1\}$

$f(0) = f(1) = 0$

$\triangleright$ $\dot{s} = 0$

⑤



▷ UW-FV: Non-physical expansion shock

11.3.4.4 Godunov Flux

$$F_{GW}(v,w) = f(u^{\downarrow}(v,w)), \quad u^{\downarrow}(v,w) = \begin{cases} v, & \dot{s} \geq 0 \\ w, & \dot{s} < 0 \end{cases}$$

$\uparrow$ intermediate state

▷ $u^{\downarrow}(v,w)$ will always corresponding to the state at $x=0$ of a weak shock solution of a R.P. with initial states $u_l = v$, $u_r = w$

▷ F_{UW} wrong for transonic rarefactions!

Idea: obtain suitable intermediate state as

$$u^{\downarrow}(v,w) = \psi(0)\,, \qquad (11.3.4.27)$$

where $\overline{u}(x,t) = \psi(x/t)$ solves the Riemann problem ($\rightarrow$ Def. 8.2.5)

$\hookrightarrow$ cf. similarity solution

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial x} f(\overline{u}) = 0\,, \quad \overline{u}(x,0) = \begin{cases} v & \text{, for } x < 0\,, \\ w & \text{, for } x \geq 0\,. \end{cases} \qquad (11.3.4.28)$$

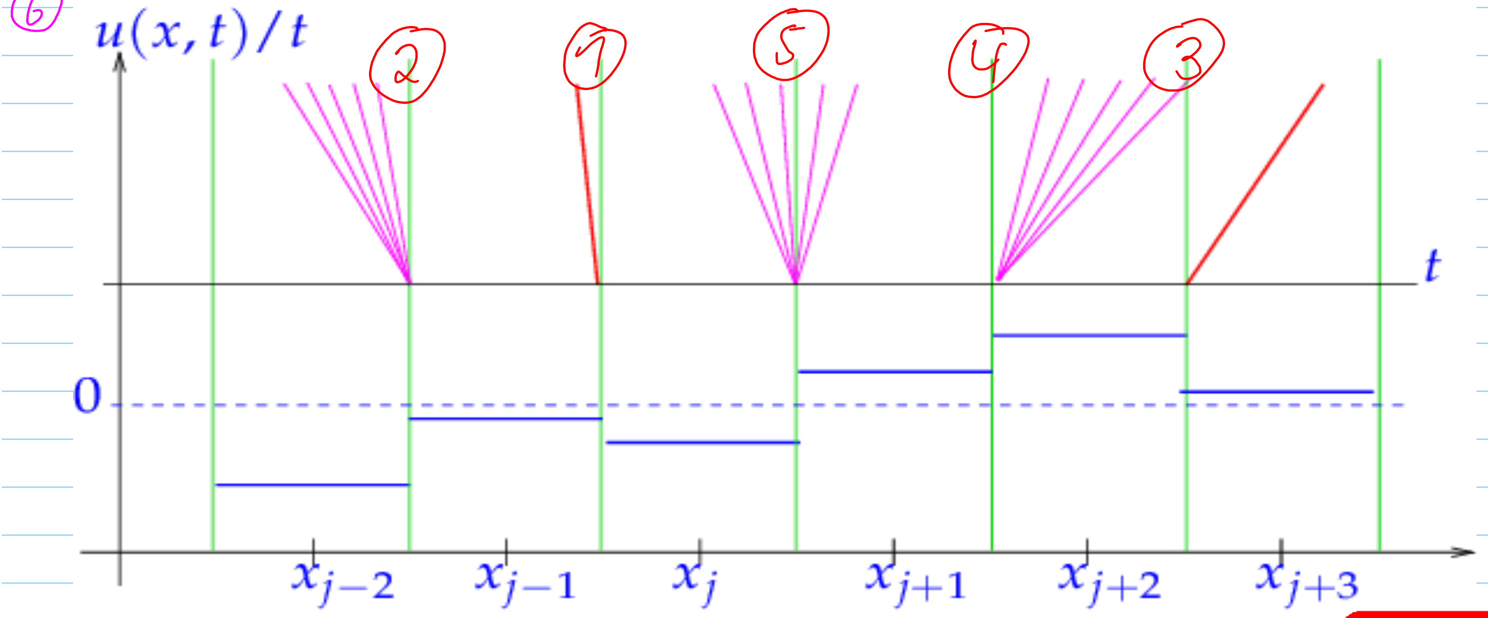
Discussion for convex f (f' increasing)

❶ If $v > w$ ➤ discontinuous solution, shock ($\rightarrow$ Lemma 8.2.5.4)

$$\overline{u}(t,x) = \begin{cases} v & \text{if } x < \dot{s}t\,, \\ w & \text{if } x > \dot{s}t\,, \end{cases} \qquad \dot{s} = \frac{f(v)-f(w)}{v-w}\,. \qquad (11.3.4.29)$$

❷ If $v \leq w$ ➤ continuous solution, rarefaction wave ($\rightarrow$ Lemma 8.2.5.10)

$$\overline{u}(t,x) = \begin{cases} v & \text{if } x < f'(v)t\,, \\ g(x/t) & \text{if } f'(v) \leq x/t \leq f'(w)\,, \\ w & \text{if } x > f'(w)t\,, \end{cases} \qquad g := (f')^{-1}\,. \qquad (11.3.4.30)$$



$$u^{\downarrow}(v,w) = \begin{cases} w & \text{, if } \begin{array}{l} v > w \wedge \dot{s} < 0 \text{ (shock 1)}, \\ v < w \wedge f'(w) < 0 \text{ (rarefaction 2)}, \end{array} \\ v & \text{, if } \begin{array}{l} v > w \wedge \dot{s} > 0 \text{ (shock 3)}, \\ v < w \wedge f'(v) > 0 \text{ (rarefaction 4)}, \end{array} \\ (f')^{-1}(0) & \text{, if } v < w \wedge f'(v) \leq 0 \leq f'(w) \text{ (rarefaction 5)}. \end{cases} \qquad (11.3.4.31)$$

transonic rarefaction

▷ Godunov numerical flux

$$F_{GD}(v,w) = f(u^{\downarrow}(v,w))$$

$\neq F_{LW}$ only in the case of transonic rarefaction

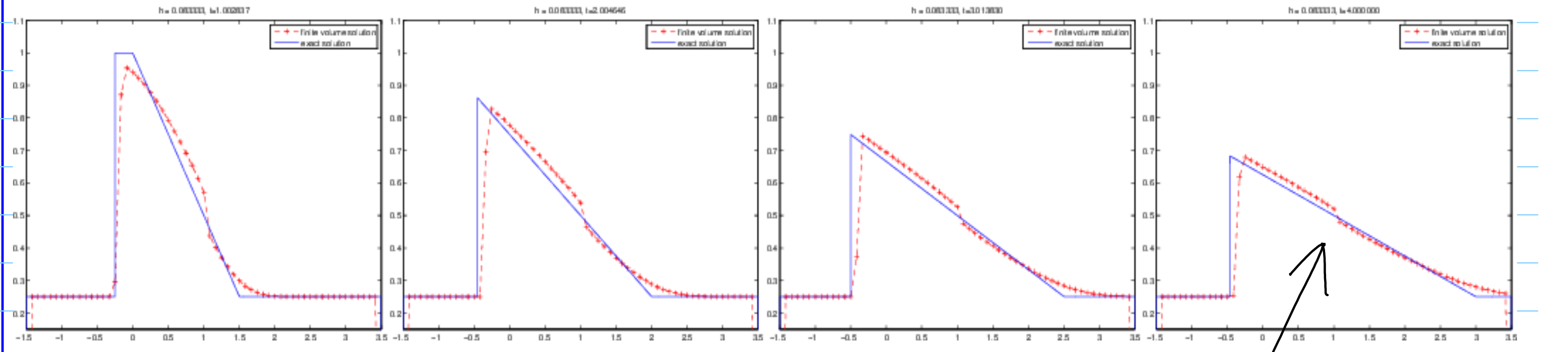
Final formula:

▶ Godunov numerical flux function

$$F_{GD}(v,w) = \begin{cases} \min_{v \leq u \leq w} f(u) & \text{, if } v < w , \\ \max_{w \leq u \leq v} f(u) & \text{, if } w \leq v . \end{cases} \qquad (11.3.4.33)$$

Exp 11.3.4.36:

F_{GD} & traffic flow : transonic rarefaction



▷ correct rarefaction fan

remnant of expansion shock at $u = \frac{1}{2}$, $f'(\frac{1}{2}) = 0$

⑦ # Review questions 11.3.4.37

[Answer without aids]

A:

State concrete formulas for the following two-point numerical flux functions, when applied to a 1D scalar conservation law with flux function

1. $f(u) = \exp(u)$, $u \in \mathbb{R}$,
2. $f(u) = u^2$, $u \in \mathbb{R}$:

❶ Central flux

$$F_C(v,w) = f(\tfrac{1}{2}(v+w)) \;, \quad v,w \in \mathbb{R} \;,$$

❷ (local) Lax-Friedrichs flux:

$$F_{LF}(v,w) = \tfrac{1}{2}(f(v)+f(w)) - \tfrac{1}{2}(w-v) \cdot \max_{\min\{v,w\} \le u \le \max\{v,w\}} |f'(u)| \;, \tag{8.3.4.16}$$

❸ upwind flux

$$F_{uw}(v,w) = \begin{cases} f(v) & \text{, if } \dot{s} \ge 0 \;, \\ f(w) & \text{, if } \dot{s} < 0 \;, \end{cases} \qquad \dot{s} := \begin{cases} \frac{f(w)-f(v)}{w-v} & \text{for } v \ne w \;, \\ f'(v) & \text{for } v = w \;, \end{cases} \tag{8.3.4.19}$$

❹ Godunov flux:

$$F_{GD}(v,w) = \begin{cases} \min_{v \le u \le w} f(u) & \text{, if } v < w \;, \\ \max_{w \le u \le v} f(u) & \text{, if } w \le v \;. \end{cases} \tag{8.3.4.33}$$

B:

Give an example of a strictly concave flux function $f : \mathbb{R} \to \mathbb{R}$ and of two states $v, w \in \mathbb{R}$, for which the upwind flux

$$F_{uw}(v,w) = \begin{cases} f(v) & \text{, if } \dot{s} \ge 0 \;, \\ f(w) & \text{, if } \dot{s} < 0 \;, \end{cases} \qquad \dot{s} := \begin{cases} \frac{f(w)-f(v)}{w-v} & \text{for } v \ne w \;, \\ f'(v) & \text{for } v = w \;, \end{cases} \tag{8.3.4.19}$$

and the Godunov flux:

$$F_{GD}(v,w) = \begin{cases} \min_{v \le u \le w} f(u) & \text{, if } v < w \;, \\ \max_{w \le u \le v} f(u) & \text{, if } w \le v \;, \end{cases} \tag{8.3.4.33}$$

yield different results.

Give an example of a strictly concave flux function f, for which they agree for all input states.

C:

For Burgers equation with flux function $f(u) = u^2$ characterize all pairs of left and right states $u_l, u_r \in \mathbb{R}$ for which a conservative finite volume discretization of the 1D Riemann problem based on the upwind flux produces an expansion shock that violates the Lax entropy condition.

Hint. The upwind flux fails in situations where transonic rarefactions emerge as entropy solutions.