ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 11.3.5: Monotone Schemes

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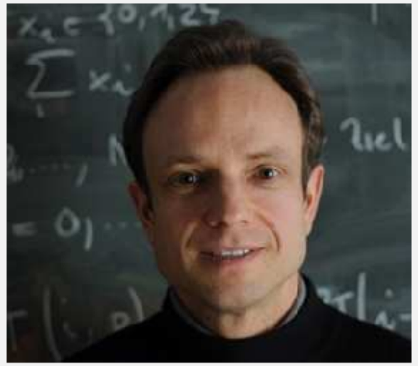


Dependency. [Lecture → Section 11.3.2] and [Lecture → Section 11.3.4], also related to [Lecture → Section 11.2.7]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

XI Numerical Methods for Conservation Laws

11.3 Conservative Finite-Volume (FV) Discretization

11.3.5 Monotone Finite-Volume Schemes

FV-MOL-ODE: $\dot{\mu}_j(t) = -\frac{1}{h}(F(\mu_j,\mu_{j+1}) - F(\mu_{j-1},\mu_j))$

Concern: Structure preservation

Key structural properties of entropy solutions

(A) $\min_{x\in\mathbb{R}} u_0(x) \leq u(x,t) \leq \max_{x\in\mathbb{R}} u_0(x)$

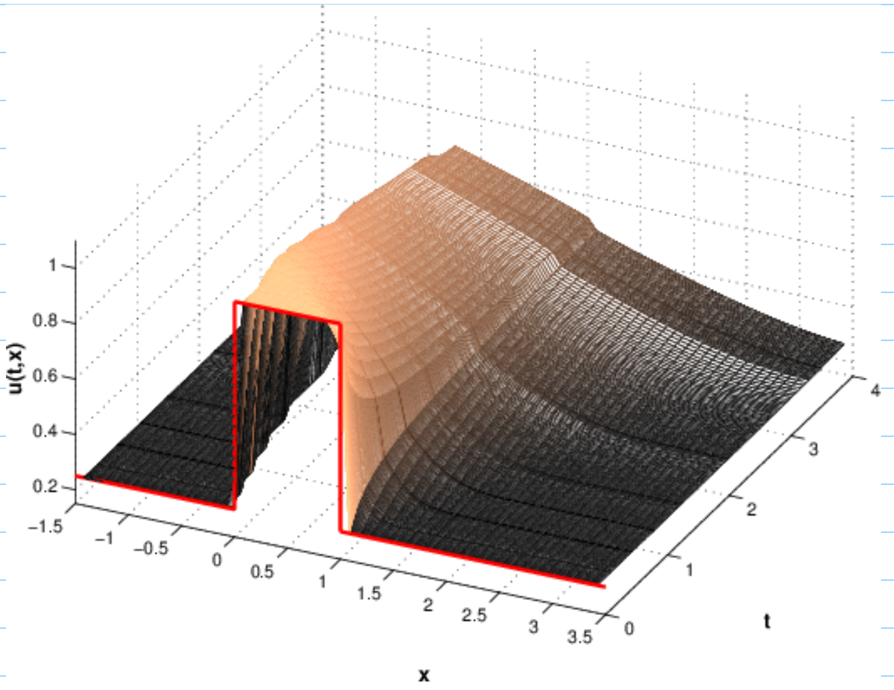
Theorem 11.2.7.1. Comparison principle for scalar conservation laws

If $u_0 \leq \bar{u}_0$ a.e. on $\mathbb{R}$ $\Rightarrow$ $u \leq \bar{u}$ a.e. on $\mathbb{R}\times]0,T[$

(B) "no new extrema"

u solves (8.2.2.1) ➤ No. of local extrema (in space) of $u(\cdot,t)$ decreasing with time

Exp: F_{GD} & traffic flow



FV-solution satisfies (A) & (B)

(8.3.2.7) ➤ $$\frac{d\mu_j}{dt}(t) = -\frac{1}{h}\left(F(\mu_j(t),\mu_{j+1}(t)) - F(\mu_{j-1}(t),\mu_j(t))\right), \quad j\in\mathbb{Z}, \qquad (8.3.2.8)\ (*)$$

for Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad \text{in } \mathbb{R}\times]0,T[\;,\quad u(x,0)=u_0(x)\,,\quad x\in\mathbb{R}, \qquad (8.2.2.1)$$

Goal: Semi-discrete comparison principle

$$\left.\begin{array}{l}\vec\eta(t),\ \vec\mu(t) \text{ solve } (*)\\ \eta_j(0)\le\mu_j(0)\end{array}\right\} \Rightarrow \begin{array}{l}\eta_j(t)\le\mu_j(t)\\ \forall j\in\mathbb{Z},\ \forall t\ge 0\end{array}$$

Assume: $\vec\eta$ catches up with $\vec\mu$ at $t=t^*$, & node x_j

$$\Rightarrow \left\{\begin{array}{l}\eta_k(t)\le\mu_k(t),\quad \forall k\in\mathbb{Z},\ \forall t\le t^*\\ \eta_j(t^*)=\mu_j(t^*)=\xi\end{array}\right.$$

$$(*) \triangleright \frac{d}{dt}(\mu_j(t)-\eta_j(t)) = -\frac{1}{h}\left(F(\xi,\mu_{j+1})-F(\xi,\eta_{j+1}) + F(\eta_{j-1},\xi)-F(\mu_{j-1},\xi)\right)$$

$\eta_j(t)$ will never manage to overtake $\mu_j(t)$, if $\frac{d}{dt}(\mu_j(t)-\eta_j(t)) \ge 0$:

Satisfied, if
$$\begin{array}{l}F(\xi,\mu_{j+1})-F(\xi,\eta_{j+1})\le 0\\ F(\eta_{j-1},\xi)-F(\mu_{j-1},\xi)\le 0\end{array} \qquad (2)$$

whenever $\mu_{j+1}\ge\eta_{j+1}$, $\mu_{j-1}\ge\eta_{j-1}$

③

(2) satisfied, if

$v \to F(v,w)$ increasing $\forall w$

$w \to F(v,w)$ decreasing $\forall v$

FEC? $\Longleftrightarrow$ $\frac{\partial F}{\partial v}(v,w) \geq 0$, $\frac{\partial F}{\partial w}(v,w) \leq 0$ $\forall (v,w)$

Definition 8.3.5.5. Monotone numerical flux function

A 2-point numerical flux function $F = F(v,w)$ is called monotone, if

F is an increasing function of its first argument v $(\forall w)$

and

F is a decreasing function of its second argument w $(\forall v)$.

$\hat{=}$ Sufficient condition for Semi-discrete comparison principle !

Corollary 8.3.5.6. Simple criterion for monotone flux function

A continuously differentiable 2-point numerical flux function $F = F(v,w)$ is montone, if and only if

$$\frac{\partial F}{\partial v}(v,w) \geq 0 \quad \text{and} \quad \frac{\partial F}{\partial w}(v,w) \leq 0 \quad \forall (v,w)\,. \tag{8.3.5.7}$$

Lemma 8.3.5.8. Monotonicity of Lax-Friedrichs/Rusanov numerical flux and Godunov flux

For any continuously differentiable flux function f the associated Lax-Friedrichs/Rusanov flux (8.3.4.16) and Godunov flux (8.3.4.33) are monotone.

Proof for Godunov flux:

▶ Godunov numerical flux function

$$F_{GD}(v,w) = \begin{cases} \min\limits_{v \leq u \leq w} f(u) & \text{, if } v < w\,, \\ \max\limits_{w \leq u \leq v} f(u) & \text{, if } w \leq v\,. \end{cases} \tag{8.3.4.33}$$

(i) If $v < w$: $v \uparrow$ or $w \downarrow$

$\Rightarrow$ Minimum over smaller range: $F_{GD} \uparrow$

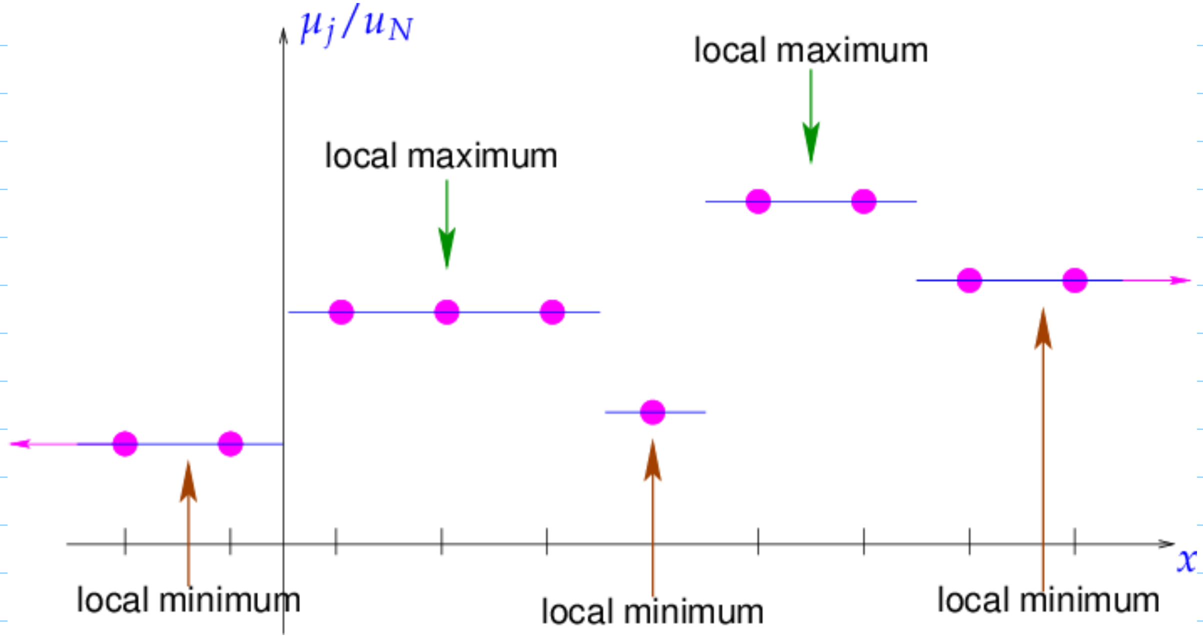
(ii) If $v \geq w$: $v \uparrow$ or $w \downarrow$

$\Rightarrow$ Maximum taken over a larger range

$\Rightarrow$ $F_{GD} \uparrow$

4

(B) No emergence of new discrete extrema



Lemma II.3.5.13. Non-oscillatory monotone semi-discrete evolutions

If $\vec{\mu} = \vec{\mu}(t)$ solves (II.3.2.8) with a monotone numerical flux function $F = F(v,w)$ and $\vec{\mu}(0)$ has finitely many local extrema, then the number of local extrema of $\vec{\mu}(t)$ cannot be larger than that of $\vec{\mu}(0)$.

Proof: $i \hat{=}$ index of a local maximum of $\vec{\mu}(t)$

local maximum property

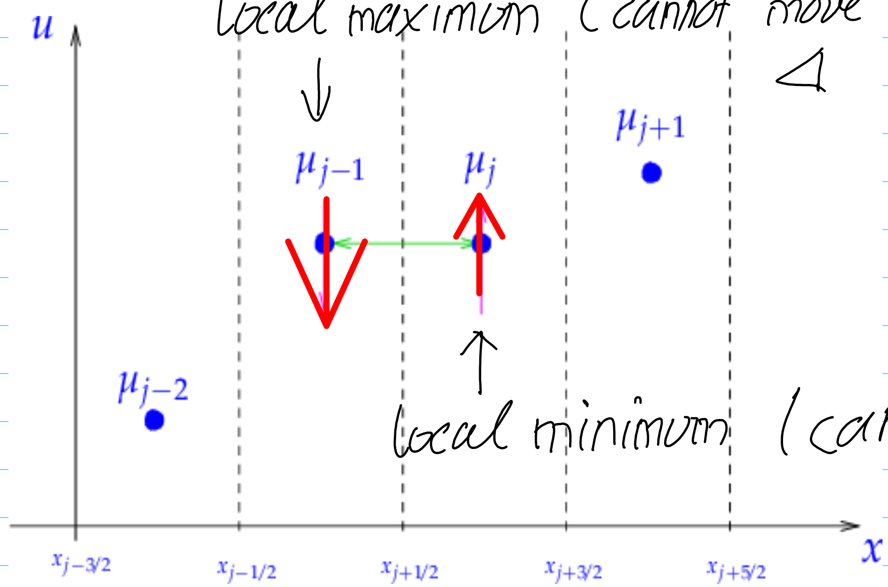
$\Downarrow$

$\mu_{i-1}(t) \leq \mu_i(t)$, $\mu_{i+1}(t) \leq \mu_i(t)$ $\overset{\text{monotone flux}}{\Longrightarrow}$ $F(\mu_i,\mu_{i+1}) \geq F(\mu_i,\mu_i) \geq F(\mu_{i-1},\mu_i)$,

$$\Rightarrow \quad \frac{d}{dt}\mu_i(t) = -\frac{1}{h}\left(F(\mu_i,\mu_{i+1}) - F(\mu_{i-1},\mu_i)\right) \leq 0\,.$$

↳ Local maximum decreases

Similar: Local minimum grows



◁ Can $\mu_{j-1}(t)$ overtake $\mu_j(t)$?

⑤ Review questions 11.3.5.14 :

[Please tackle without looking at notes]

A:

Show that for smooth and convex $f : \mathbb{R} \to \mathbb{R}$ the Godunov numerical flux

$$F_{\mathrm{GD}}(v,w) = \begin{cases} \min\limits_{v \le u \le w} f(u) & \text{, if } \quad v < w\,, \\ \max\limits_{w \le u \le v} f(u) & \text{, if } \quad w \le v\,, \end{cases} \tag{8.3.4.33}$$

satisfies

$$\frac{\partial F_{\mathrm{GD}}}{\partial v}(v,w) \ge 0 \quad \text{and} \quad \frac{\partial F_{\mathrm{GD}}}{\partial w}(v,w) \le 0 \quad \forall (v,w)\,, \tag{8.3.5.7}$$

$$\left|\frac{\partial F_{\mathrm{GD}}}{\partial v}(v,w)\right|, \left|\frac{\partial F_{\mathrm{GD}}}{\partial w}(v,w)\right| \le \max_{\min\{v,w\} \le u \le \max\{v,w\}} |f'(u)| \quad \forall (v,w)\,.$$

B:

A mapping $\boldsymbol{\mathcal{H}} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ ($\mathbb{R}^{\mathbb{Z}}$ is the vector space of real-valued sequences $(\mu_j)_{æ\in\mathbb{Z}}$, equivalently, the vector space of functions $\mathbb{Z} \mapsto \mathbb{R}$) is called **monotone***, if

$$\zeta_j \ge \mu_j \quad \forall j \in \mathbb{Z} \quad \Rightarrow \quad \left(\boldsymbol{\mathcal{H}}((\zeta_k)_{k\in\mathbb{Z}})\right)_j \ge \left(\boldsymbol{\mathcal{H}}((\mu_k)_{k\in\mathbb{Z}})\right)_j \quad \forall j \in \mathbb{Z}\,.$$

- Show that $\boldsymbol{\mathcal{H}} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ which is continuously differentiable in each component is monotone, if

$$\left(\frac{\partial \boldsymbol{\mathcal{H}}}{\partial \mu_k}\right)_j \ge 0 \quad \forall j,k \in \mathbb{Z}\,.$$

- Applying explicit Euler timestepping with timestep $\tau > 0$ to the semi-discrete evolution arising from the conservative finite-volume discretization of a scalar conservation law $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$ on an equidistant spatial mesh (mesh width $h > 0$) and based on the Godunov numerical flux yields a mapping $\boldsymbol{\mathcal{H}} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$. Show that this mapping is monotone provided that

$$\frac{\tau}{h} \le \frac{1}{2M} \quad , \quad M := \max\{|f'(u)| : \min_{x\in\mathbb{R}} u_0(x) \le u \le \max_{x\in\mathbb{R}} u_0(x)\}\,.$$

Hint. Depends on Question (**Q**8.3.5.14.A).

Remark. Mappings like $\boldsymbol{\mathcal{H}}$ will be investigated more closely in § 8.4.1.8.

6