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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 11.4.1–Section 11.4.2: Timestepping: Fully Discrete Evolutions and CFL-Condition

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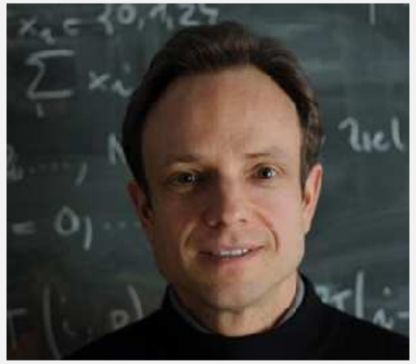


Dependency. [Lecture → Section 11.3.2]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

XI Numerical Methods for Conservation Laws

11.4 Timestepping for Finite-Volume Methods (FVM)

11.4.1 Fully Discrete Evolutions

MOL-approach: applying numerical integration scheme to FV-MOL-ODE

$$\vec{\mu}:[0,T]\to\mathbb{R}^{\mathbb{Z}}:\quad \dot{\mu}_j(t) = \overbrace{-\frac{1}{h}\left(F(\mu_j,\mu_{j+1}) - F(\mu_{j-1},\mu_j)\right)}^{=:\,(\mathcal{L}_h(\vec{\mu}))_j}, \quad j\in\mathbb{Z}$$

+ truncation to a finite set of indices

Note: $\mathcal{L}_h$ is non-linear & complicated

→ explicit timestepping scheme

② Usual choice: explicit RK-SSMs

Definition 7.3.3.1. General Runge-Kutta method →

For coefficients $b_i, a_{ij} \in \mathbb{R}$, $c_i := \sum_{j=1}^{s} a_{ij}$, $i,j = 1,\dots,s$, $s \in \mathbb{N}$, the discrete evolution $\Psi^{s,t}$ of an s-stage Runge-Kutta single step method (RK-SSM) for the ODE $\dot{\mathbf{u}} = \mathbf{f}(t,\mathbf{u})$, is defined by

$$\mathbf{k}_i := \mathbf{f}(t + c_i\tau, \mathbf{u} + \tau \sum_{j=1}^{s} a_{ij}\mathbf{k}_j)\,, \quad i = 1,\dots,s \quad , \quad \Psi^{t,t+\tau}\mathbf{u} := \mathbf{u} + \tau \sum_{i=1}^{s} b_i \mathbf{k}_i\,.$$

The $\mathbf{k}_i \in V_0$ are called increments.

Butcher scheme

$$\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^T \end{array} \;\hat{=}\; \begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & \ddots & & a_{2s} \\ \vdots & \vdots & & \ddots & \vdots \\ c_s & a_{s1} & \cdots & & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}\,, \quad \mathbf{c},\mathbf{b} \in \mathbb{R}^s,\quad \mathfrak{A} \in \mathbb{R}^{s,s}\,. \tag{7.3.3.3}$$

explicit RK-SSM: $a_{ij} = 0$ for $j \geq i$
($\mathcal{A}$ strictly lower-triangular)

Apply to $\dot{\vec{\mu}} = \mathcal{L}_h(\vec{\mu})$, timestep $\tau > 0$

▷ for s-stage RK-SSM

$$\vec{\kappa}_1 = \mathcal{L}_h(\vec{\mu}^{(k)})\,,$$
$$\vec{\kappa}_2 = \mathcal{L}_h(\vec{\mu}^{(k)} + \tau a_{21}\vec{\kappa}_1)\,,$$
$$\vec{\kappa}_3 = \mathcal{L}_h(\vec{\mu}^{(k)} + \tau a_{31}\vec{\kappa}_1 + \tau a_{32}\vec{\kappa}_2)\,,$$
$$\vdots$$
$$\vec{\kappa}_s = \mathcal{L}_h(\vec{\mu}^{(k)} + \tau \sum_{j=1}^{s-1} a_{sj}\vec{\kappa}_j)\,,$$

↑ increments

$$\underbrace{\vec{\mu}^{(k+1)} = \vec{\mu}^{(k)} + \tau \sum_{l=1}^{s} b_l \vec{\kappa}_j\,.}_{\text{update formula}} \tag{11.4.1.7}$$

▷ Fully discrete evolution
$\vec{\mu}^{(k+1)} = \mathcal{H}_h(\vec{\mu}^{(k)})$
⇕
approximation of $u(x,t)$: $\mu_j^{(k)} \approx u(jh, k\tau)$
(on uniform space-time grid)

Examples: for 2-point numerical flux $F = F(v,w)$

• Explicit Euler $\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$:

(8.3.2.8) ➤ $$(\mathcal{L}_h\vec{\mu})_j := -\frac{1}{h}\big(F(\mu_j,\mu_{j+1}) - F(\mu_{j-1},\mu_j)\big)\,, \quad \vec{\mu} \in \mathbb{R}^{\mathbb{Z}},\ j \in \mathbb{Z}\,. \tag{11.4.1.5}$$

explicit Euler

$$\longrightarrow \quad \vec{\mu}^{(k+1)} = \vec{\mu}^{(k)} + \tau\mathcal{L}_h(\vec{\mu}^{(k)})\,.$$

▶ $$\boxed{(\mathcal{H}_h(\vec{\mu}))_j = \mu_j - \frac{\tau}{h}\big(F(\mu_j,\mu_{j+1}) - F(\mu_{j-1},\mu_j)\big)}\,. \tag{11.4.1.10}$$

(3) Explicit trapezoidal rule :
$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

explicit trapezoidal rule

$$\vec{\kappa} = \mu^{(k)} + \tau \mathcal{L}_h(\vec{\mu}^{(k)}) \quad , \quad \vec{\mu}^{(k+1)} = \mu^{(k)} + \frac{\tau}{2}\left(\mathcal{L}_h(\mu^{(k)}) + \mathcal{L}_h(\vec{\kappa})\right) .$$

▶ $$\begin{cases} \kappa_j := (\vec{\kappa})_j = \mu_j - \frac{\tau}{h}\left(F(\mu_j, \mu_{j+1}) - F(\mu_{j-1}, \mu_j)\right) , \\ (\mathcal{H}_h(\vec{\mu}))_j = \mu_j - \frac{\tau}{2h}\left(F(\kappa_j, \kappa_{j+1}) - F(\kappa_{j-1}, \kappa_j) + F(\mu_j, \mu_{j+1}) - F(\mu_{j-1}, \mu_j)\right) . \end{cases} \tag{11.4.1.11}$$

↳ ... ↳ depends on $\mu_{j-1}, \mu_{j-1}, \mu_j$

Explicit midpoint rule
$$\begin{array}{c|cc} \frac{1}{2} & 0 & 0 \\ & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

explicit midpoint rule

$$\rightarrow \quad \vec{\kappa} = \mu^{(k)} + \frac{\tau}{2}\mathcal{L}_h(\vec{\mu}^{(k)}) \quad , \quad \vec{\mu}^{(k+1)} = \mu^{(k)} + \tau \mathcal{L}_h(\vec{\kappa}) .$$

▶ $$\begin{cases} \kappa_j := (\vec{\kappa})_j = \mu_j - \frac{\tau}{2h}\left(F(\mu_j, \mu_{j+1}) - F(\mu_{j-1}, \mu_j)\right) , \\ (\mathcal{H}_h(\vec{\mu}))_j = \mu_j - \frac{\tau}{h}\left(F(\kappa_j, \kappa_{j+1}) - F(\kappa_{j-1}, \kappa_j)\right) . \end{cases} \tag{11.4.1.12}$$

11.4.2 CFL – Condition

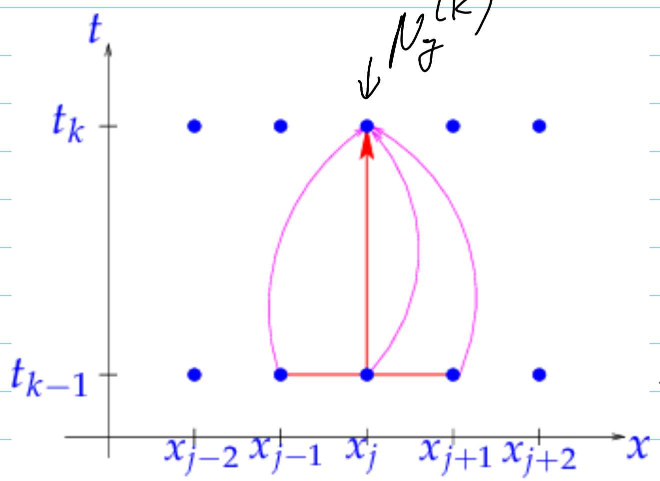
▷ Stability induced timestep constraint on τ in terms of h.

$$\vec{\mu}^{(k+1)} = \mathcal{H}_h(\vec{\mu}^{(k)}) , \quad k \in \mathbb{N}_0$$

↑ local operator

Visualization of flow of information effected by $\mathcal{H}_h$ on space-time grid

$\mu_j^{(k)}$

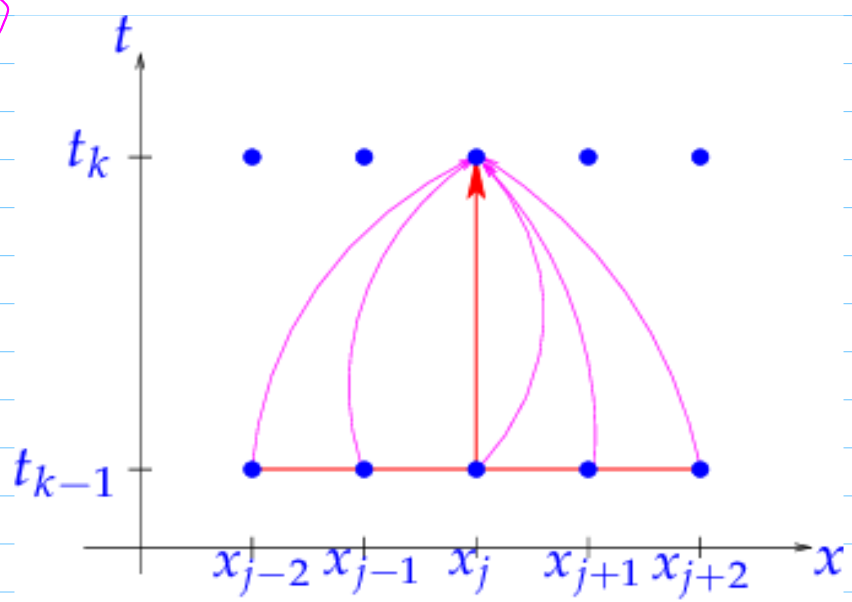


2-point numerical flux & explicit Euler timestepping

◁ $(\mathcal{H}_h(\vec{\mu}))_j = \mu_j - \frac{\tau}{h}\left(F(\mu_j, \mu_{j+1}) - F(\mu_{j-1}, \mu_j)\right)$

◁ Stencil notation

◁ Stencil width $m = 1$



2-point numerical flux & explicit trapezoidal rule

$\triangleleft$ Stencil width $m = 2$

Summary: properties of $\mathcal{H}_h$
[for multi-point numerical fluxes]

(I) $\mathcal{H}_h$ is local

$$(\mathcal{H}_h(\vec{\mu}))_j = \mathcal{H}_j(\mu_{j-m_\ell}, \dots, \mu_{j+m_r}),\ j \in \mathbb{Z}$$

for some $m_\ell, m_r \in \mathbb{N}_0$

$$(\mathcal{L}_h(\vec{\mu}))_j = \mathcal{L}(\mu_{j-m}, \dots, \mu_{j+m})$$

& s-stage RK-SSM

$\triangleright$ $m_\ell, m_r \leq s \cdot m$ ("stencil spreading")

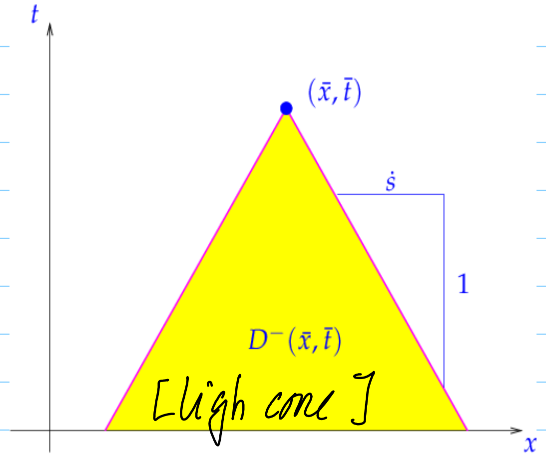
(II) If $f = f(u)$

$\Rightarrow$ $\mathcal{H}_h$ translation-invariant

$$(\mathcal{H}_h(\vec{\mu}))_j = \mathcal{H}(\mu_{j-m_\ell}, \dots, \mu_{j-m_r})$$

$\uparrow$ independent of j

Recall: analytic domain of dependence



$D^-(\bar{x}, \bar{t}) \subset \mathbb{R} \times [0, T]$

From Thm. 8.2.7.3 recall the maximal analytical domain of dependence for a solution of (8.2.2.1)

$$D^-(\overline{x}, \overline{t}) := \{(x,t) \in \mathbb{R} \times [0, \overline{t}] \colon \dot{s}_{\min}(\overline{t} - t) \leq x - \overline{x} \leq \dot{s}_{\max}(\overline{t} - t)\}\,.$$

with maximal speeds of propagation

maximal slopes of characteristics

$$\dot{s}_{\min} := \min\{f'(\xi) \colon \underbrace{\inf_{x\in\mathbb{R}} u_0(x) \leq \xi \leq \sup_{x\in\mathbb{R}} u_0(x)}_{\text{range for } u}\}\,, \qquad (11.4.28)$$

$$\dot{s}_{\max} := \max\{f'(\xi) \colon \inf_{x\in\mathbb{R}} u_0(x) \leq \xi \leq \sup_{x\in\mathbb{R}} u_0(x)\}\,. \qquad (11.4.29)$$

(5)

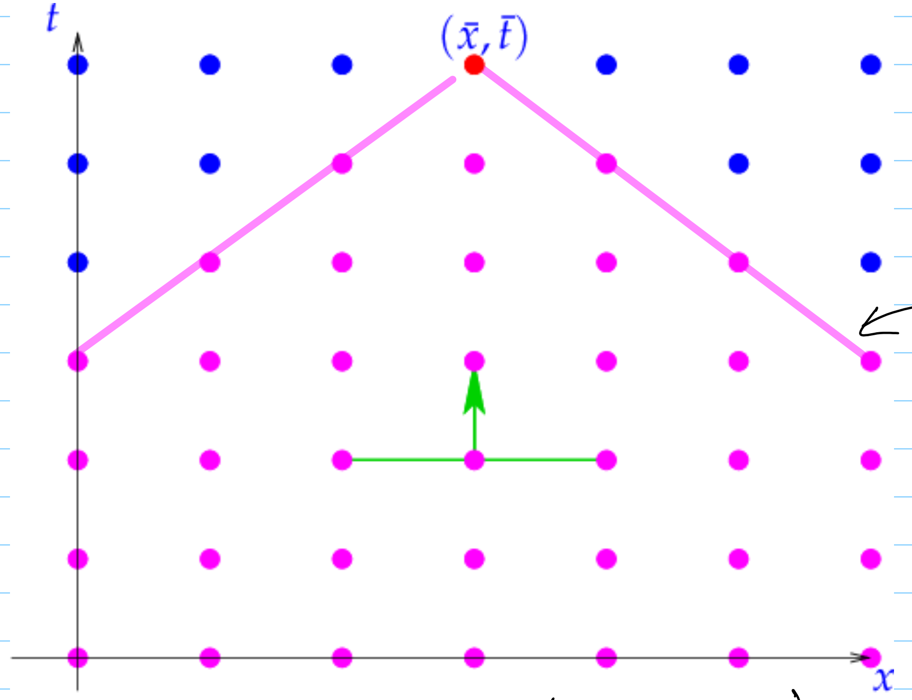
Definition |[.4.2.5. Numerical domain of dependence

Consider explicit translation-invariant fully discrete evolution $\vec{\mu}^{(k+1)} := \mathcal{H}(\vec{\mu}^{(k)})$ on uniform spatio-temporal mesh ($x_j = hj$, $j \in \mathbb{Z}$, $t_k = k\tau$, $k \in \mathbb{N}_0$) with

$$\exists m \in \mathbb{N}_0: \quad (\mathcal{H}(\vec{\mu}))_j = \mathcal{H}(\mu_{j-m}, \dots, \mu_{j+m}) \,, \quad j \in \mathbb{Z} \,. \qquad (|[4.2.6)$$

Then the numerical domain of dependence is given by

$$D_h^-(x_j, t_k) := \{(x_m, t_l) \in \mathbb{R} \times [0, t_k]: j - m(k-l) \le m \le j + m(k-l)\} \,.$$



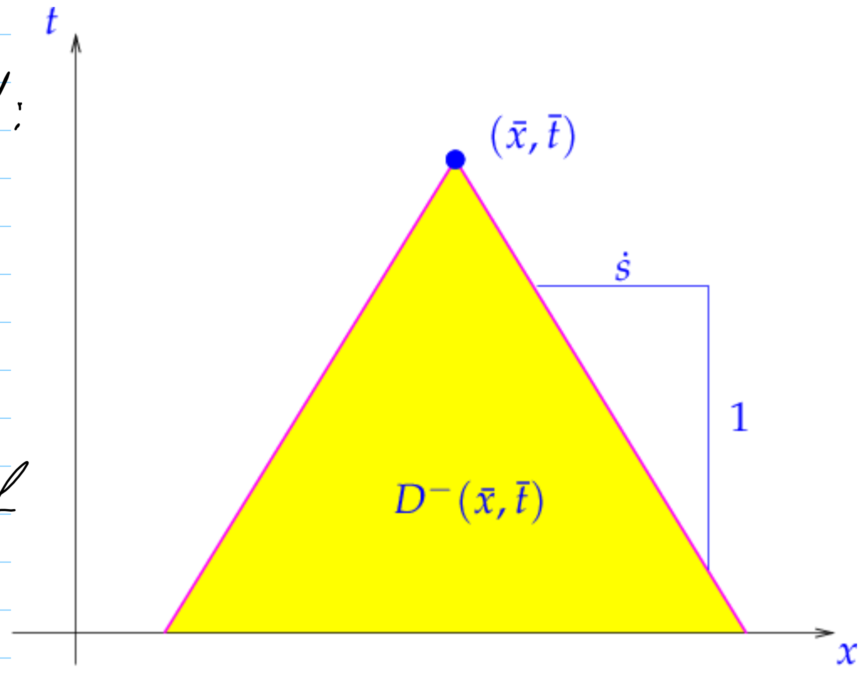
$\bullet \;\hat{=}$ numerical DoD

← slop $\frac{\tau}{h}$

$D_h^-(\bar{x}, \bar{t})$ for 3-point stencil $(m = 1)$

$$\boxed{\text{CFL-condition}: \; D^-(x_j, t_k) \subset \text{convex}\{D_h^-(x_j, t_k)\}}$$

symmetric stencil stencil, width m:



$D^-(\bar{x}, \bar{t}) \subset \mathbb{R} \times [0, T]$

Sufficient for CFL-cond:

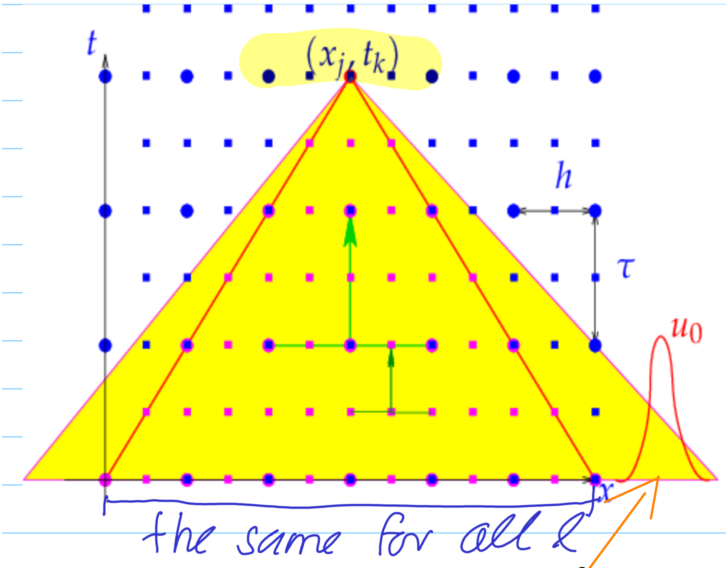
$$\frac{1}{m} \frac{\tau}{h} \le \frac{1}{\dot{s}}$$

$\dot{s} \;\hat{=}$ maximal speed of propagation

$\hat{=}$ timestep constraint

$$\tau \le m \, h / \dot{s}$$

CFL-condition necessary for convergence for $\tau, h \to 0$



(● ≙ coarse grid, ■ ≙ fine grid, ■ ≙ d.o.d)

◁ Sequence of equidistant space-time grids of $\mathbb{R} \times [0,T]$ with $\tau = \gamma h$ (τ/h = meshwidth in time/space)

If $\gamma >$ CFL-constraint (8.4.2.11) then

analytical domain of dependence $\not\subset$ numerical domain of dependence

Sequence of uniform space-time grids $\mathcal{G}_\ell$, refinement level $\ell \in \mathbb{N}_0$:

$$h = 2^{-\ell} h_0 \quad , \quad \tau = 2^{-\ell}\tau_0$$

Grid point $(x_j, t_k) \in \mathcal{G}_\ell$ with $j = 2^\ell j_0$, $k = 2^\ell k_0$

$\longleftrightarrow$ same point in x-t-plane ,

If D^- (yellow ↑) strictly larger than D_h^-

u_0 supported inside D^-, but outside D_h^-

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

$u(x_j, t_k) \neq 0$ possible $\qquad\qquad$ $\mu_j^{(k)} = 0 \quad \forall \ell$

⑦ Review questions 11.4.2.13 :

Supplied information :

Definition 6.2.6.31. General Runge-Kutta method →

For coefficients $b_i, a_{ij} \in \mathbb{R}$, $c_i := \sum_{j=1}^{s} a_{ij}$, $i,j = 1,\dots,s$, $s \in \mathbb{N}$, the discrete evolution $\Psi^{s,t}$ of an **s-stage Runge-Kutta single step method** (RK-SSM) for the ODE $\dot{\mathbf{u}} = \mathbf{f}(t,\mathbf{u})$, is defined by

$$\mathbf{k}_i := \mathbf{f}(t + c_i\tau, \mathbf{u} + \tau\sum_{j=1}^{s} a_{ij}\mathbf{k}_j)\ , \quad i = 1,\dots,s \quad , \quad \Psi^{t,t+\tau}\mathbf{u} := \mathbf{u} + \tau\sum_{i=1}^{s} b_i\mathbf{k}_i\ .$$

The $\mathbf{k}_i \in V_0$ are called **increments**.

Butcher scheme :

$$\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^T \end{array} \quad \hat{=} \quad \begin{array}{c|ccccc} c_1 & a_{11} & a_{12} & \cdots & \cdots & a_{1s} \\ c_2 & a_{21} & \ddots & & & a_{2s} \\ \vdots & \vdots & & \ddots & & \vdots \\ c_s & a_{s1} & \cdots & & & a_{ss} \\ \hline & b_1 & b_2 & \cdots & \cdots & b_s \end{array} \quad , \quad \mathbf{c}, \mathbf{b} \in \mathbb{R}^s, \quad \mathfrak{A} \in \mathbb{R}^{s,s}\ . \tag{6.2.6.33}$$

A

We consider an abstract semi-discrete evolution in $\mathbb{R}^{\mathbb{Z}}$:

$$\frac{d\vec{\mu}}{dt}(t) = \mathcal{L}_h(\vec{\mu}(t))\ , \quad \mathcal{L}_h : \mathbb{R}^{\mathbb{Z}} \mapsto \mathbb{R}^{\mathbb{Z}}\ .$$

We perform timestepping based on an s-stage RK-SSM described by the following butcher scheme:

$$\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^T \end{array} \quad \hat{=} \quad \begin{array}{c|cccccc} 0 & 0 & \cdots & & & \cdots & 0 \\ c_2 & a_{21} & \ddots & & & & \vdots \\ \vdots & 0 & & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ c_s & 0 & \cdots & \cdots & 0 & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \cdots & & \cdots & b_s \end{array} \quad , \quad \mathbf{c}, \mathbf{b} \in \mathbb{R}^s, \quad \mathfrak{A} \in \mathbb{R}^{s,s}\ .$$

Elaborate the fully discrete evolution equations.

B :

The 2-step Adams-Bashforth **multi-step** method with uniform timestep $\tau > 0$ applied to the ODE $\dot{\mathbf{u}} = \mathbf{F}(t,\mathbf{u})$ generates a sequence ~~of sequence~~ of approximate states $\mathbf{u}^{(j)} \approx \mathbf{u}(t_j)$, $t_j := t_0 + \tau j$, by the **3-term recursion**

$$\mathbf{u}^{(j+1)} = \mathbf{u}^{(j)} + \tau\left(\tfrac{3}{2}\mathbf{f}(t_j, \mathbf{u}^{(j)}) - \tfrac{1}{2}\mathbf{f}(t_{j-1}, \mathbf{u}^{(j-1)})\right)\ . \tag{8.4.2.14}$$

Give the stencil of the fully discrete evolution when this timestepping scheme is used for the FV-MOL-ODE

$$\frac{d\mu_j}{dt}(t) = -\frac{1}{h}\left(F(\mu_j(t), \mu_{j+1}(t)) - F(\mu_{j-1}(t), \mu_j(t))\right)\ , \quad j \in \mathbb{Z}\ , \tag{8.3.2.8}$$

on a uniform space-time grid and with some numerical flux function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

C:

We consider a Cauchy problem for 1D linear advection with $v > 0$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(vu) = 0 \quad \text{in} \quad \widetilde{\Omega} = \mathbb{R}\times]0,T[\,, \quad u(x,0) = u_0(x) \quad \forall x \in \mathbb{R}\,. \tag{8.1.1.11}$$

For spatial discretization we use a conservative finite volume method with Lax-Friedrichs/Rusanov numerical flux. For timestepping we employ

1. the explicit Euler single-step method,
2. the implicit Euler single-step method.

In both cases derive the equations for the resulting fully discrete evolutions on a uniform space-time grid with spatial mesh width $h > 0$ and timestep $\tau > 0$.
For both choices characterize the numerical domain of dependence for $v = 1$.

Hint. The (local) Lax-Friedrichs/Rusanov numerical flux function for the scalar conservation law $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$ is

$$F_{\mathrm{LF}}(v,w) = \tfrac{1}{2}(f(v) + f(w)) - \tfrac{1}{2}(w - v) \cdot \max_{\min\{v,w\} \le u \le \max\{v,w\}} |f'(u)|\,. \tag{8.3.4.16}$$

D:

Give reasons why explicit timestepping methods are preferred for the semi-discrete evolution problems arising from conservative spatial finite-volume discretization of 1D scalar conservations laws.

E:

We consider the fully discrete evolution for solving a Cauchy problem for a 1D scalar conservation law based on

(i) a conservative finite-volumne spatial discretization with 2-point numerical flux,
(ii) an s-stage explicit Runge-Kutta single step method described by the Butcher scheme

$$\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^T \end{array} \quad \hat{=} \quad \begin{array}{c|cccc} 0 & 0 & \cdots & \cdots & 0 \\ c_2 & a_{21} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \cdots & b_s \end{array} \,, \quad \mathbf{c}, \mathbf{b} \in \mathbb{R}^s, \quad \mathfrak{A} \in \mathbb{R}^{s,s}\,.$$

We obtain a solution $\vec{\mu}^{(k)} \in \mathbb{R}^{\mathbb{Z}}$, $k \in \mathbb{N}_0$.
Assume that the initial data $u_0 : \mathbb{R} \to \mathbb{R}$ are constant outside a bounded interval. Which conditions on the RK-SSM ensure the discrete conservation property

$$\sum_{j\in\mathbb{Z}} \mu_j^{(k)} = \sum_{j\in\mathbb{Z}} \mu_j^{(0)} \quad \forall k \in \mathbb{N}_0\,, \quad \mu_j^{(k)} := \left(\vec{\mu}^{(k)}\right)_j \,?$$