ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 11.4.3: Timestepping: Linear Stability Analysis

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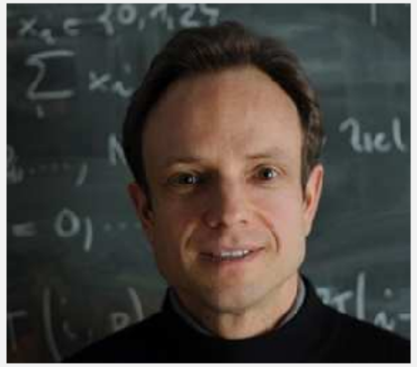


Prerequisites.
- Fourier series

Dependency. [Lecture → Section 11.3.2]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

XI. Numerical Methods for Conservation Laws

11.4. Timestepping for Finite-Volume Methods (FVM)

11.4.3. Linear Stability Analysis (von Neumann)

↔ diagonalization technique

Focus: C.P. for linear advection w/ constant velocity

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0 \quad \text{in } \mathbb{R}\times]0,T[\,, \quad u(x,0) = u_0(x) \quad \forall x \in \mathbb{R}\,. \qquad (11.1.1.11)$$

$[v > 0]$

FV spatial semi-discretization on uniform spatial grid.

FV-MOL-ODE : $\dot{\vec{\mu}}(t) = \mathcal{L}_h(\vec{\mu}(t))$, $\vec{\mu}(t) \in \mathbb{R}^{\mathbb{Z}}$

linear, local, translation-invariant

➢ *linear, local, and translation-invariant* semi-discrete evolution

$$\frac{d\vec{\mu}}{dt}(t) = \mathcal{L}_h(\vec{\mu}(t)) \,, \quad \text{with} \quad \boxed{(\mathcal{L}_h(\vec{\mu}))_j = \sum_{l=-m}^{m} c_l \mu_{j+l} \,, \quad j \in \mathbb{Z}} \,, \tag{II.4.3.2}$$

↳ Stencil operator, width m

Example : Godunov / upwind flux : $F(u_l, u_r) = v u_l$

▷ $(\mathcal{L}_h(\vec{\mu}))_j = -\frac{v}{h}(\mu_j - \mu_{j-1})$

(II.4.3.2) with $c_0 = -v/h$, $c_{-1} = v/h$

Needed : eigenvectors of $\mathcal{L}_h$, guess inspired by translational invariance

Try : $\vec{\psi}_\xi := [\exp(\imath \xi j)]_{j\in\mathbb{Z}}$, $-\pi < \xi \leq \pi$

"Fourier mode" $\quad$ ↑ frequency parameter

$$(\mathcal{L}_h(\vec{\psi}_\xi))_j = \sum_{l=-m}^{m} c_l e^{\imath\xi(j+l)} = \underbrace{\left(\sum_{l=-m}^{m} c_l e^{\imath\xi l}\right)}_{\in\,\mathbb{C}\,:\ \text{eigenvalue of } \mathcal{L}_h} \underbrace{e^{\imath\xi j}}_{=(\vec{\psi}_\xi)_j}$$

$\hat{c}(\xi) := \left(\sum_{l=-m}^{m} c_l e^{\imath\xi l}\right) : [-\pi,\pi] \to \mathbb{C}$: symbol

Example : Upwind flux $\quad c_0 = -v/h$, $c_{-1} = v/h$

▷ $\hat{c}_{uw}(\xi) = v/h\,(e^{\imath\xi} - 1)$

Central flux : $c_{-1} = \frac{v}{2h}$, $c_1 = -\frac{v}{2h}$

▷ $\hat{c}_c(\xi) = -\imath \frac{v}{h} \sin(\xi)$

Apply to explicit Euler timestepping :

$$\vec{\mu}^{(k+1)} = \vec{\mu}^{(k)} + \tau \mathcal{L}_h(\vec{\mu}^{(k+1)})$$

$\vec{\mu}^{(0)} = \vec{\psi}_\xi \Rightarrow \vec{\mu}^{(1)} = (1 + \tau\hat{c}(\xi))\,\vec{\psi}_\xi$

$\Rightarrow \vec{\mu}^{(k)} = (1+\tau\hat{c}(\xi))^k\,\vec{\psi}_\xi$

$\vec{\mu}^{(k)}$ bounded $\Leftrightarrow$ $|1+\tau\hat{c}(\xi)|\leq 1$

stable $\Leftrightarrow$ $|1+\tau\hat{c}(\xi)|\leq 1 \quad \forall -\pi<\xi\leq\pi$

For general RK-SSM:

$$\vec{\mu}^{(0)} = \vec{\Psi}_\xi \Rightarrow \boxed{\vec{\mu}^{(k)} = S(\tau\hat{c}(\xi))^k\, \vec{\Psi}_\xi}$$

stability function of RK-SSM

$S:\mathbb{C}\to\mathbb{C}$ defined by applying RK-SSM to $\dot{y}=\lambda y$: $y^{(1)} = S(\tau\lambda)\, y^{(0)}$

Expl. Euler: $S(z) = 1 + z$

Stability of RK-timestepping of linear semi-discrete evolution $\Longleftrightarrow$ $\max_{-\pi<\xi\leq\pi} |S(\tau\hat{c}(\xi))|\leq 1$

Example: Upwind flux & expl. Euler

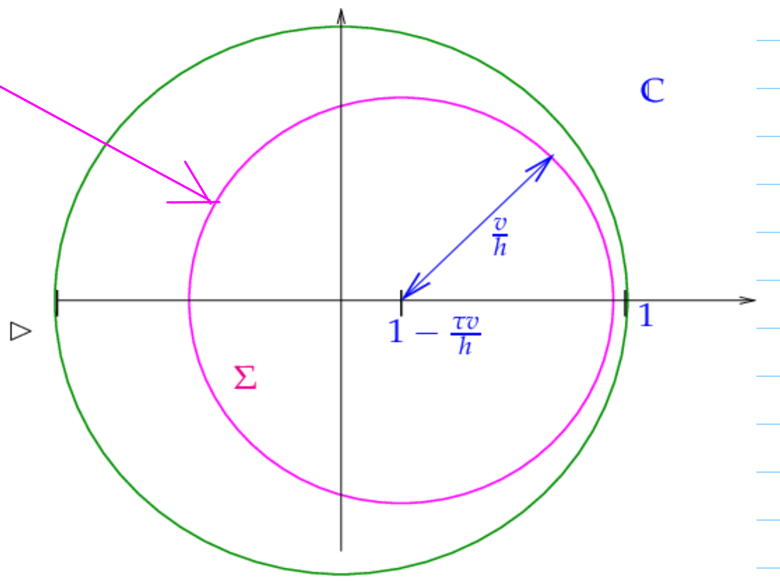
$$S(\tau\hat{c}(\xi)) = 1 - v\tau/h + v\tau/h\, e^{i\xi}, \quad \pi<\xi\leq\pi$$

Locus of

$\Sigma := S(\tau\hat{c}(\xi))$, $-\pi<\xi\leq\pi$,

in the complex plane

(Unit circle in green) ▷



$|S(\tau c(\xi))|\leq 1 \quad \forall\xi$

$\Leftrightarrow v\tau/h \leq 1$

$\Leftrightarrow \tau \leq h/v$

[same as CFL-condition, since $v = \dot{\xi}$]

Example: Central flux & expl. Euler

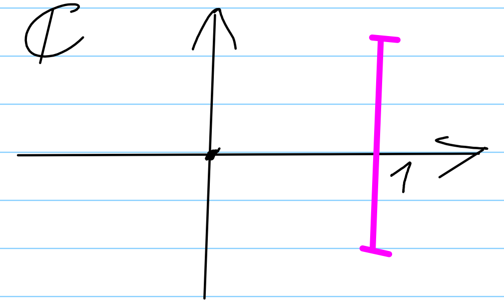
$$F(\mu_l,\mu_r) = \tfrac{1}{2}v(\mu_l+\mu_r)$$

$$(\mathcal{L}_h\vec{\mu})_j = -v/h\,(\mu_{j+1}-\mu_{j-1})$$

(4)

▷ $S(\tau\hat{c}(\xi)) = 1 + v\,{}^{\tau}\!/_{h}\, 2i\sin(\xi)$

ℂ



$|S(\tau c(\xi))| > 1$
$\forall \xi \neq 0$

Unconditional instability : blow-up $\forall \tau > 0$

Theorem ||.4.3.20. Stability function of explicit Runge-Kutta methods

The execution of one step of size $\tau > 0$ *of an explicit* s*-stage Runge-Kutta single step method (→ Def. 6.2.6.32) with Butcher scheme* $\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^T \end{array}$ *(see (6.2.6.34)) for the scalar linear ODE* $\dot{y} = \lambda y$*,* $\lambda \in \mathbb{C}$*, amounts to a multiplication with the number*

$$\Psi_\lambda^\tau = \underbrace{1 + z\mathbf{b}^T(\mathbf{I} - z\mathfrak{A})^{-1}\mathbf{1}}_{\text{stability function } S(z)} = \det(\mathbf{I} - z\mathfrak{A} + z\mathbf{1}\mathbf{b}^T)\,, \quad z := \lambda\tau\,, \quad \mathbf{1} = (1,\dots,1)^T \in \mathbb{R}^s\,.$$

$S \in \mathcal{P}_s$: a polynomial

$D_s := \{ z \in \mathbb{C} : |S(z)| \leq 1 \}$ is bounded

≙ stability domain

$S(\tau\hat{c}(\xi)) \in D_s \;\; \forall \xi$ only if τ sufficiently small

≙ timestep constraint

Why is linear advection relevant

$$\partial_t u + \partial_x f(u) = \partial_t u + f'(u)\,\partial_x u = 0$$

$$\Updownarrow$$

$$\partial_t u + c\,\partial_x u = 0$$

⑤ Review questions 11.4.3.25:

A:

Let $\mathcal{L}_h : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ be a linear translation-invariant operator.

1. What does it mean that $\mathcal{L}_h$ is linear?
2. Give a definition of what is meant by "translation-invariant".
3. What is the symbol of $\mathcal{L}_h$?

B:

Conduct a von Neumann stability analysis for the linear evolution

$$\dot{\mu}_j = \frac{\mu_{j+1} - 2\mu_j + \mu_{j-1}}{h^2}\,, \quad h > 0\,, \tag{8.4.3.26}$$

when

1. explicit Euler timestepping,
2. the explicit trapezoidal rule

with uniform timestep $\tau > 0$ is used for discretization in time.

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