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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 11.4.4: Convergence of Fully Discrete FV Method

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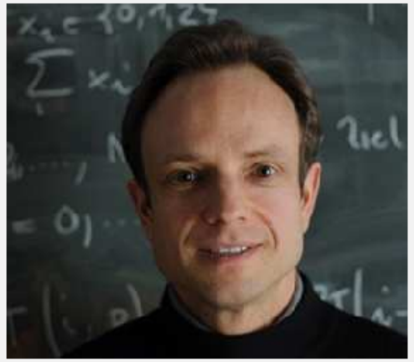


Prerequisites.
- Norms on function spaces

Dependency. [Lecture → Section 11.3.2]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

XI Numerical Methods for Conservation Laws

11.4 Timestepping for Finite-Volume Methods (FVM)

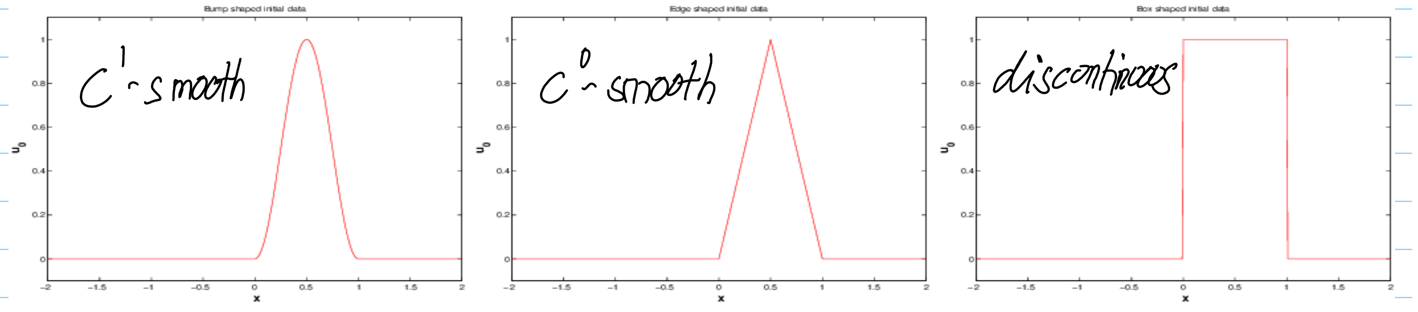
11.4.4 Convergence of Fully Discrete FVMs

↳ asymptotic perspective as $h, \tau \to 0$
[linked by CFL-condition]

Numerical experiment: C.P. for Burgers equ.:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2\right) = 0 \quad \text{in } \mathbb{R}\times]0,T[\,, \quad u(x,0) = u_0(x)\,, \quad x \in \mathbb{R}\,.$$

u_0:



(BUMP) (WEDGE) (BOX)

(2)

- Local Lax-Friedrichs / Godunov numerical fluxes
- Explicit RK-SSM (4th order)
- CFL-condition: $\tau/h = 1$

Space-time error norms:

$$\mathrm{err}_1(h) := \max_{k>0} h \sum_j |\mu_j^{(k)} - u(x_j,t_k)| \approx \max_{k>0} \left\| u_N^{(k)} - u(\cdot,t_k) \right\|_{L^1(\mathbb{R})}, \quad \text{"}L^1\text{-}L^\infty\text{"} \tag{8.4.4.2}$$

$$\mathrm{err}_\infty(h) := \max_{k>0} \max_{j\in\mathbb{Z}} |\mu_j^{(k)} - u(x_j,t_k)| \approx \max_{k>0} \left\| u_N^{(k)} - u(\cdot,t_k) \right\|_{L^\infty(\mathbb{R})}. \quad \text{"}L^\infty\text{-}L^\infty\text{"} \tag{8.4.4.3}$$

$$\|v\|_{L^1(\Omega)} = \int_\Omega |v(x)|\,dx$$

Why these norms?

→ Entropy solution stable in these norms, e.g.

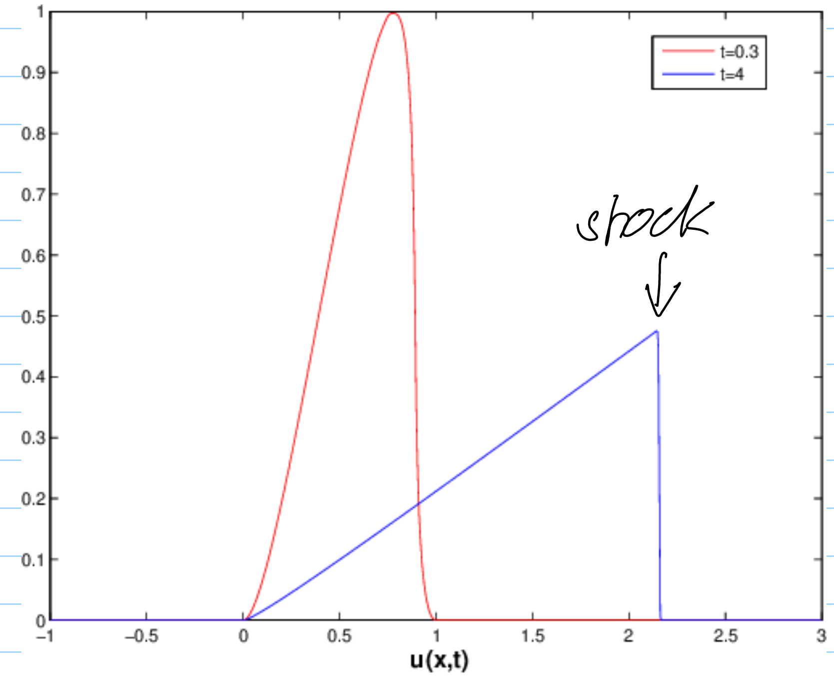
Theorem ||.2.7.3. L^1-contractivity of evolution for scalar conservation law

$$\forall t \in]0,T[,\ R>0: \quad \int_{|x|<R} |u(x,t)|\,\mathrm{d}x \le \int_{|x|<R+\dot{s}t} |u_0(x)|\,\mathrm{d}x\,,$$

↳ L^1-norm

with *maximal speed of propagation*

$$\dot{s} := \max\{|f'(\xi)| : \inf_{x\in\mathbb{R}} u_0(x) \le \xi \le \sup_{x\in\mathbb{R}} u_0(x)\}\,. \tag{8.2.7.4}$$



"Exact solution": Initial data (BUMP)

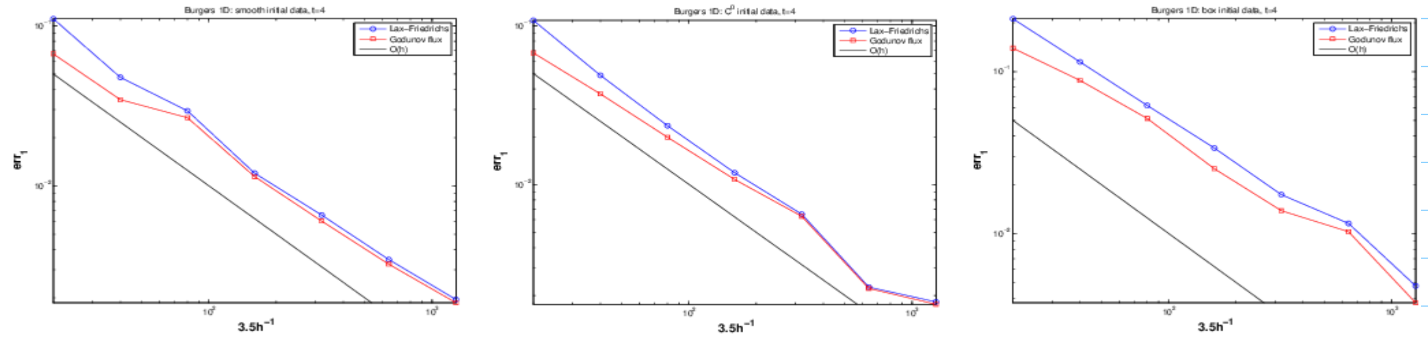
◁ "Exact solution" ≙ highly resolved numerical solution

$t = 0.3$: solution still C^1 in space

$t = 4$: shock

- L^1-L^∞-error:

$T = 0.3$, error err_1

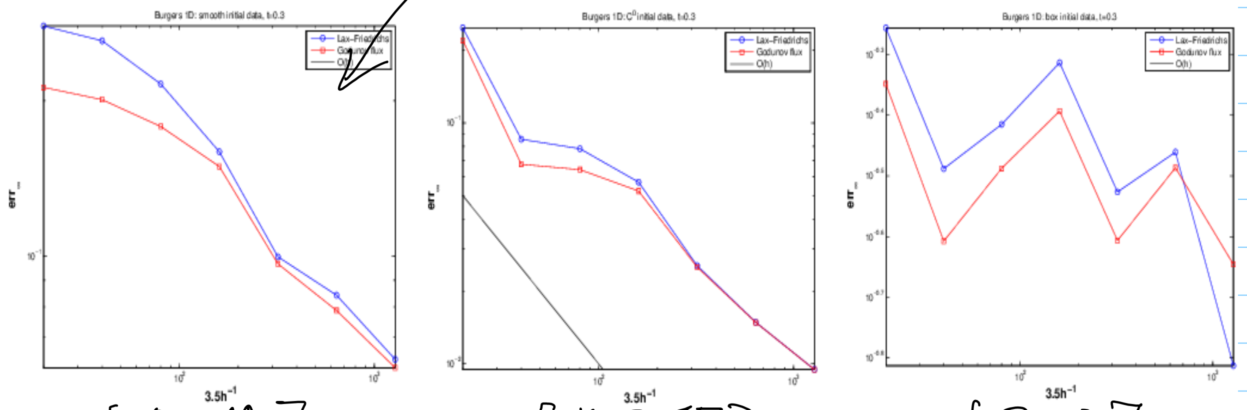


▷ Alg. convergence with rate 1: $O(h)$ as $h \to 0$

(3) L^∞-L^∞-error:

$T = 0.3$: error err_∞

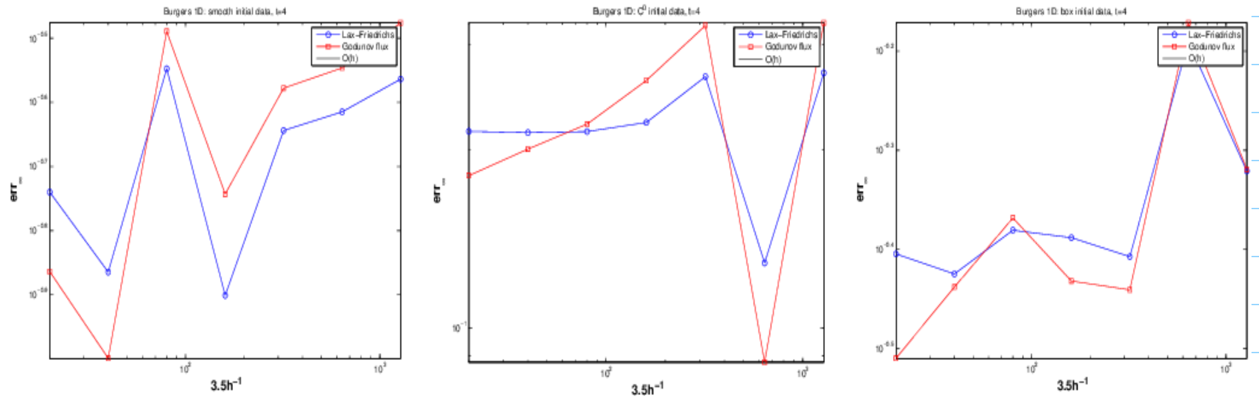
solution still smooth



[BUMP] [WEDGE] [BOX]

Converge ?

$T = 4$: error err_∞



No convergence

Observation: $O(h)$-convergence in $L^?$-L^∞, but no more

Explanation:

For a smooth (in space) solution $u = u(x,t)$ of C.P. we define the consistency error

$$\left(\vec{\tau}(t)\right)_{j+1/2} := \underbrace{F(u(x_j,t), u(x_{j+1},t))}_{\text{approximate flux}} - \underbrace{f(u(x_{j+1/2},t))}_{\text{exact flux}} \qquad (11.4.4.5)$$

$$\max_{j\in\mathbb{Z}} (\vec{\tau}_F(t))_j = O(h^q) \quad \text{for } h \to 0 \quad \leftrightarrow \quad \text{numerical flux is consistent of order } q \in \mathbb{N}\,. \qquad (8.4.4.6)$$

"Rule of thumb":

Order of alg. cvg. L^∞-L^1 $\leq$ order of consistency of numerical flux

shown by local Taylor expansion

Example: Godunov flux / upwind flux for $f'(u) \geq 0 \;\; \forall u$

$$F_{GD}(v,w) = F_{uw}(v,w) = f(v)$$

(4) Assume smooth flux function f

$$F_{uw}(u(x_j,t),u(x_{j+1},t)) = f(u(x_j),t)\,, \quad j\in\mathbb{Z}\,. \tag{II.4.4.8}$$

Taylor expansion of f around $u(x_{j+1/2},t)$)

$$(\vec{\tau}_{F_{uw}}(t))_j = f(u(x_j,t)) - f(u(x_{j+\frac{1}{2}},t)) \quad \text{[by def. of cons. err.]}$$

$$= f'(u(x_{j+\frac{1}{2}},t))(u(x_j,t) - u(x_{j+\frac{1}{2}},t)) + O(|u(x_j,t) - u(x_{j+\frac{1}{2}},t)|^2)$$

$$= -f'(u(x_{j+\frac{1}{2}},t))\frac{\partial u}{\partial x}(x_{j+\frac{1}{2}},t)\tfrac{1}{2}h + O(h^2) \quad \text{for } h\to 0\,, \quad = O(h)$$

Taylor expansion of u in space

Similar for F_{LF} : also $O(h)$

Manifestation of

Order barrier for monotone numerical fluxes

Monotone numerical fluxes (→ Def. 8.3.5.5) are at most first order consistent.

⑤ Review questions 11.4.4.12

[Try to answer without aids]

A:

When conducting convergence studies of conservative finite volume methods for Cauchy problems for scalar conservations laws on uniform space-time grids (spatial mesh width $h > 0$, timestep size $\tau > 0$) one usually examines the errors

$$\mathrm{err}_1(h) := \max_{k>0} h \sum_j |\mu_j^{(k)} - u(x_j, t_k)| \approx \max_{k>0} \left\| u_h^{(k)} - u(\cdot, t_k) \right\|_{L^1(\mathbb{R})} ,$$

$$\mathrm{err}_\infty(h) := \max_{k>0} \max_{j\in\mathbb{Z}} |\mu_j^{(k)} - u(x_j, t_k)| \approx \max_{k>0} \left\| u_h^{(k)} - u(\cdot, t_k) \right\|_{L^\infty(\mathbb{R})} .$$

1. In these formulas, what do $\mu_j^{(k)}$, t_k, u_h, and u stand for?
2. Why are the above error expressions considered relevant for scalar conservation laws?

B:

The local consistency error of a 2-point numerical flux $F = F(v, w)$ associated with the flux function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$(\vec{\tau}_F(t))_j = F(u(x_j, t), u(x_{j+1}, t)) - f(u(x_{j+\frac{1}{2}}, t)) \ , \quad j \in \mathbb{Z} \quad , \tag{8.4.4.5}$$

What is the consistency order of F, how is it computed, and what assumption limits its predictive power for the actual convergence of a conservative finite-volume method for a Cauchy problem for a scalar conservation law?