ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 11.5: Higher-Order Conservative Finite-Volume Schemes

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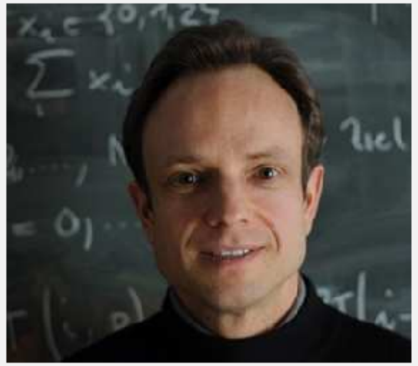


Date: May 15, 2021

Dependency. [Lecture → Section 11.3.2] [Lecture → Section 11.4.1]

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

Note the *change in chapter numbers*, which also provide leading digits for labels:
Old Chapter 6 → New Chapter 9 , Old Chapter 8 → New Chapter 11
Trailing digits in labels are not affected.

XI Numerical Methods for Conservation Laws

11.5. Higher-Order Conservative Finite Volume Schemes

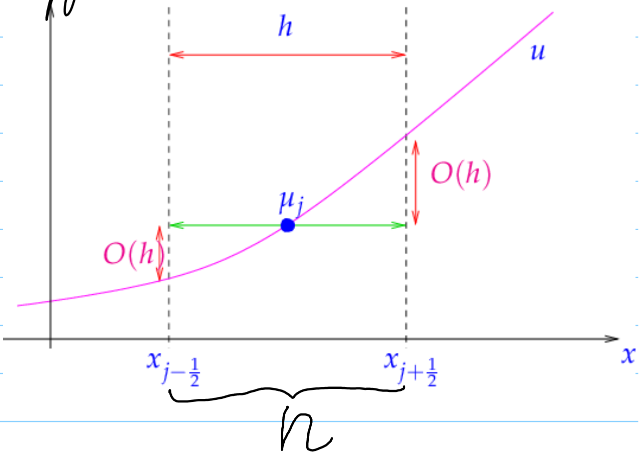
Second-order alg. cvg. as $h,\tau \to 0$
[for smooth solutions only]

Focus: Spatial consistency error

A limitation of 2-point numerical fluxes based on cell averages ≙ p.w. constant approximation

smooth

$$u(x_{j+\frac{1}{2}},t) - \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t)\,\mathrm{d}x = O(h) \quad \text{for } h \to 0\,,$$



⇒ 1st-order consistence

Exception: central flux

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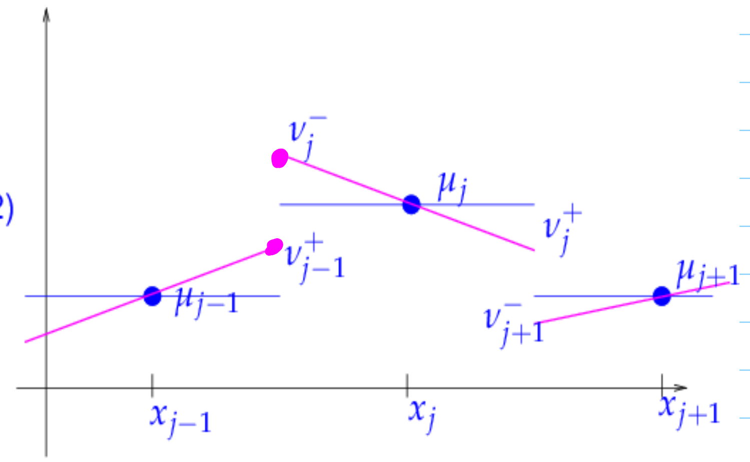
Idea : 11.5.1 Piecewise Linear Reconstruction

$\mu_h \hat{=}$ p.w. linear on dual cells

$$v_j^-(t) := \mu_j(t) - \tfrac{1}{2}h\sigma_j(t)\,,\qquad v_j^+(t) := \mu_j(t) + \tfrac{1}{2}h\sigma_j(t)\,,\qquad j \in \mathbb{Z}\,,\qquad (11.5.1.2)$$

with suitable slopes $\sigma_j(t) = \sigma(\vec{\mu}(t))$.



FV-MOL-ODE :

$$\dot{\mu}_j = -\tfrac{1}{h}\left(F(v_j^+, v_{j+1}^-) - F(v_{j-1}^+, v_j^-)\right)$$

FVM : still retain cell averages as unknowns

We need way to obtain slopes σ_j from μ_j so that $|v_j^+ - u(x_{j+1/2}, t)| = O(h^2)$ for $h \to 0$

based $\mu_{j-1}, \mu_j, \mu_{j+1}$

Definition 11.5.1.3. Linear reconstruction

Given an (infinite) mesh $\mathcal{M} := \{]x_{j-1}, x_j[\}_{j\in\mathbb{Z}}$ ($x_{j-1} < x_j$), a linear reconstruction operator $\mathsf{R}_\mathcal{M}$ is a mapping

$$\mathsf{R}_\mathcal{M} : \mathbb{R}^\mathbb{Z} \mapsto \{v \in L^\infty(\mathbb{R}) : v \text{ linear on }]x_{j-1/2}, x_{j+1/2}[\ \forall j \in \mathbb{Z}\}\,,$$

taking a sequence $\vec{\mu} \in \mathbb{R}^\mathbb{Z}$ of cell averages to a possibly *discontinuous* function $\mathsf{R}_\mathcal{M}\vec{\mu}$ that is piecewise linear on dual cells.

delivers slopes : $\sigma_j = (\mathsf{R}_\mathcal{M}\vec{\mu})'(x_j)$

Natural choice : slope by (central) difference quotient ("central slope")

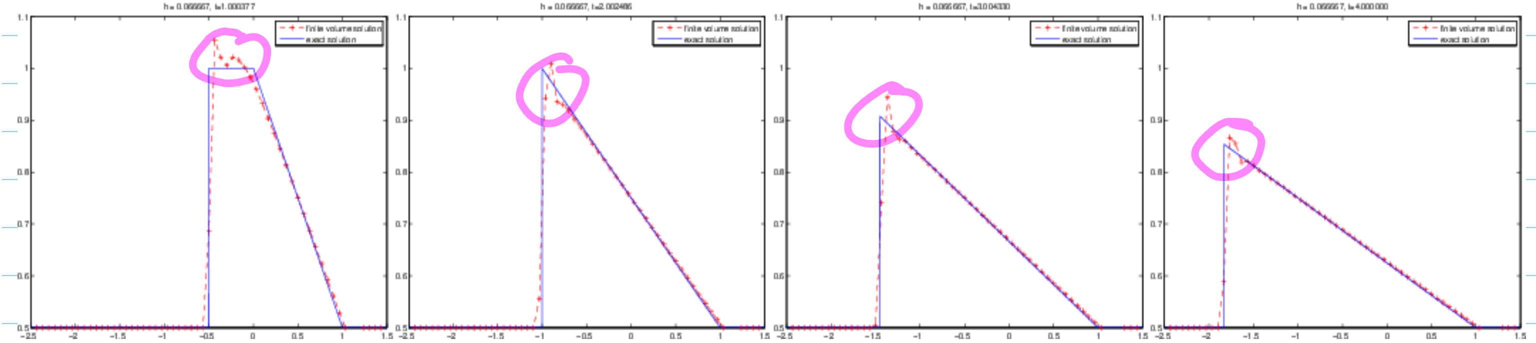
$$\sigma_j(t) = \frac{1}{2}\left(\frac{\mu_{j+1}(t) - \mu_j(t)}{h} + \frac{\mu_j(t) - \mu_{j-1}(t)}{h}\right) = \frac{1}{2}\,\frac{\mu_{j+1}(t) - \mu_{j-1}(t)}{h}\,. \qquad (11.5.1.7)$$

▷ Experiments : $O(h^2)$ alg. cvg. L^1-L^∞ error norm, if solution is smooth

Experiment : C.P. for traffic flow,

$$u_0(x) = \begin{cases} 1\,; & 0 \le x \le 1 \\ 1/2\,; & \text{elsewhere} \end{cases}$$

Godunov flux



▷ Spurious oscillations at shock

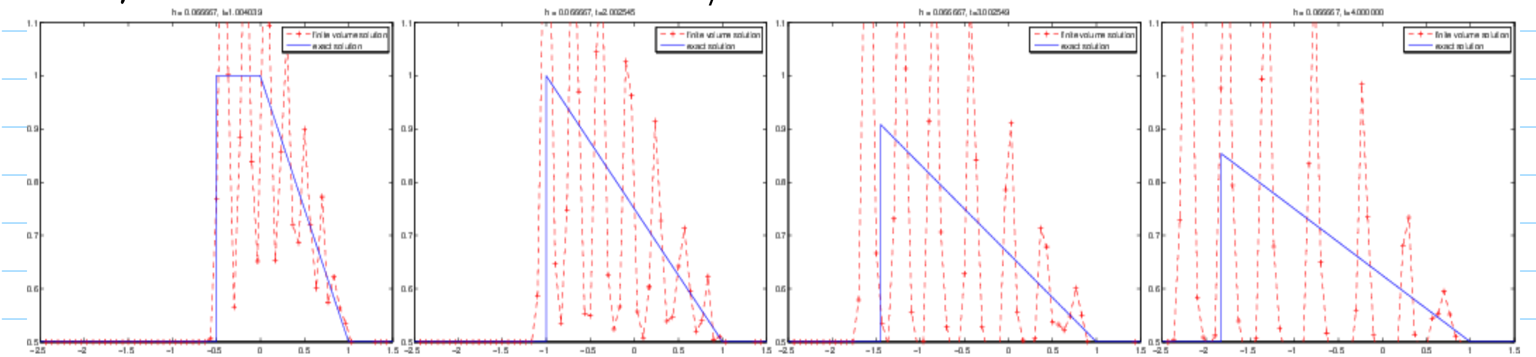
Upwind-inspired remedy?

$$\text{Right slope:}\quad \sigma_j(t) = \frac{\mu_{j+1}(t) - \mu_j(t)}{h}\,, \tag{II.5.1.12}$$

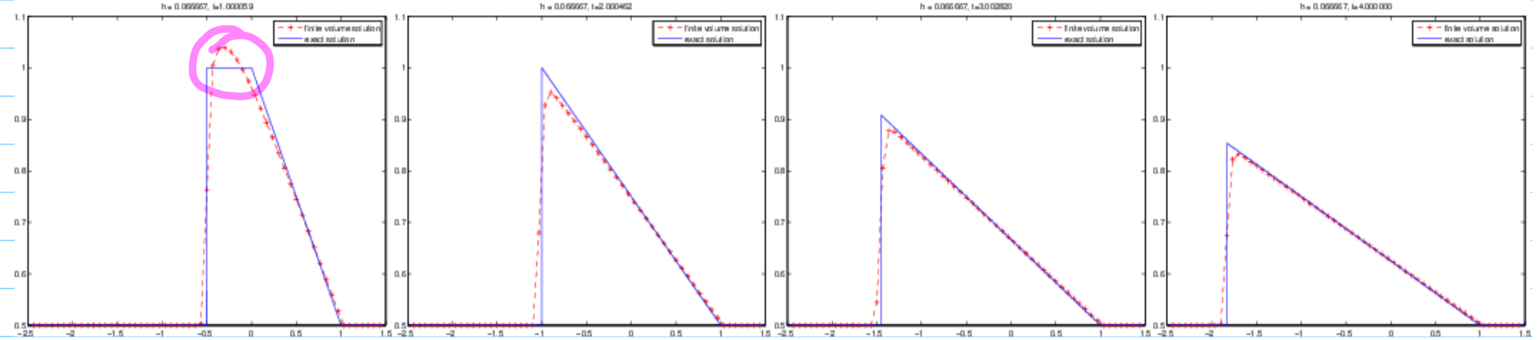
$$\text{Left slope:}\quad \sigma_j(t) = \frac{\mu_j(t) - \mu_{j-1}(t)}{h}\,. \tag{II.5.1.13}$$

↑ one-sided difference quotients

Traffic flow num. exp.:

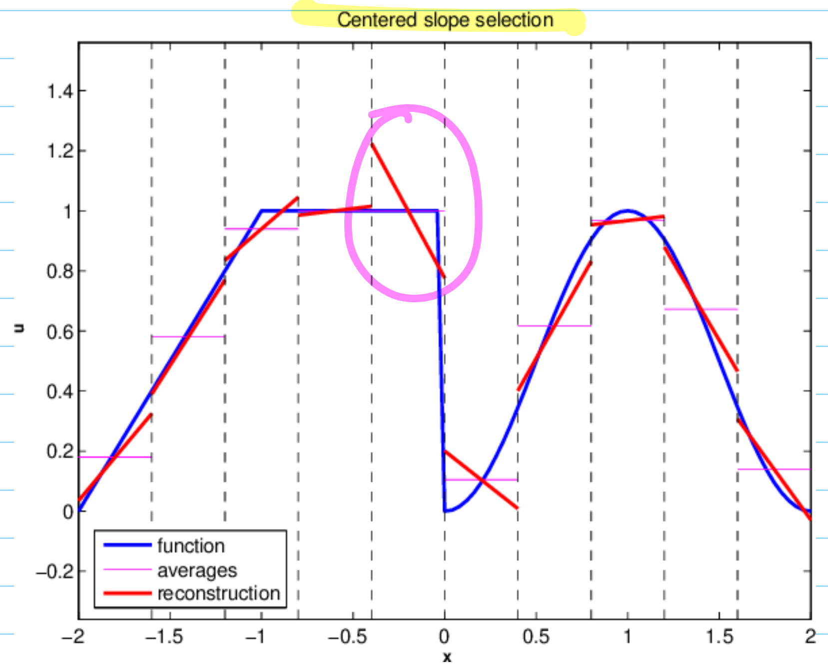


▷ blow-up

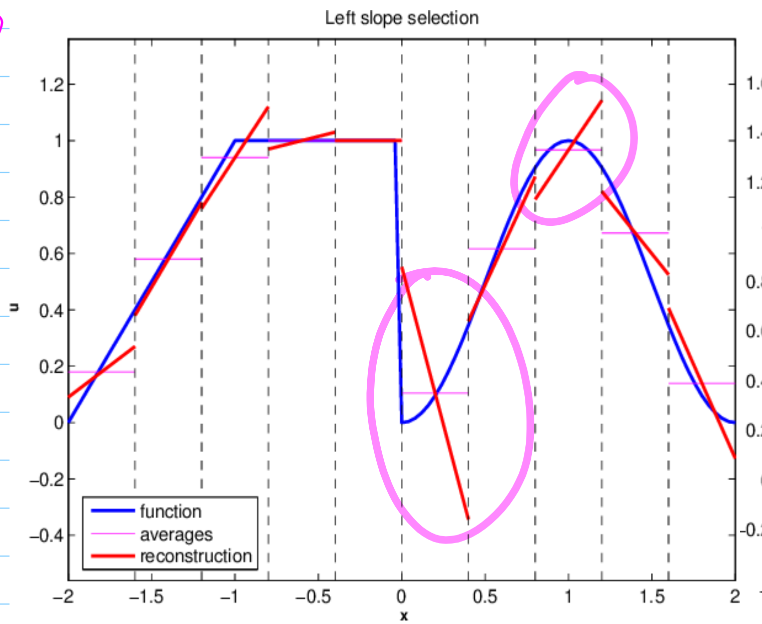


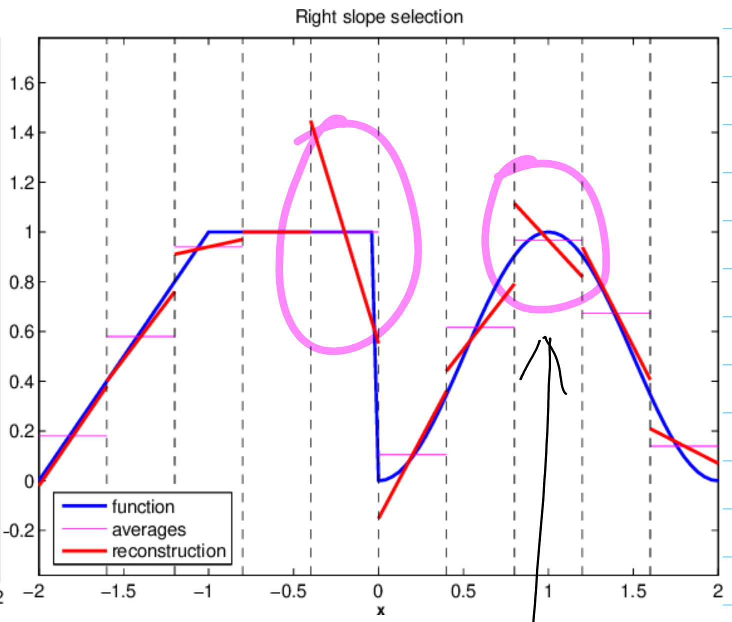
▷ Overshoots near shocks

Insight by examining p.w. linear reconstruction in a special situation

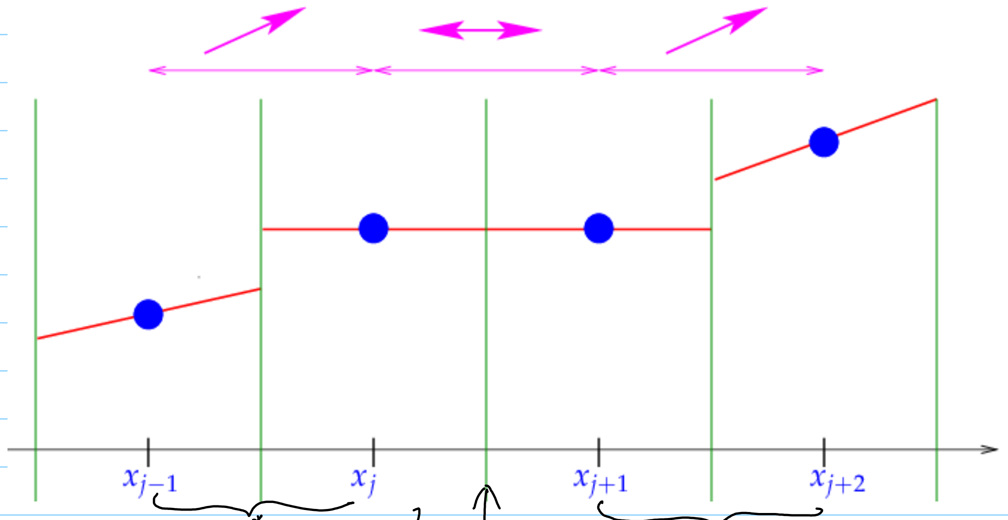


◁ overshoots at shocks





overshoot even where u is smooth



R_M increasing

slope $= 0$

Monotonicity preserving linear reconstruction:

- constant at plateaus
- constant at (local) extrema

II.5.2. Slope Limiting

Idea: To avoid over/undershoots use local monotonicity-preserving p.w. linear reconstruction

Definition II.5.2.1. Monotonicity-preserving linear reconstruction (MPLR)

An linear reconstruction operator $R_{\mathcal{M}}$ ($\to$ Def. 8.5.1.3) is monotonicity preserving, if

$$(R_{\mathcal{M}}\vec{\mu})(x_j) = \mu_j \quad \wedge \quad \begin{array}{lll} \mu_j \le \mu_{j+1} & \Rightarrow & R_{\mathcal{M}}\vec{\mu} \text{ non-decreasing in }]x_j, x_{j+1}[\,, \\ \mu_j \ge \mu_{j+1} & \Rightarrow & R_{\mathcal{M}}\vec{\mu} \text{ non-increasing in }]x_j, x_{j+1}[\,. \end{array}$$

Benefits of monotonicity preservation

- $\min_j \mu_j \le R_M\vec{\mu}(x) \le \max_j \mu_j$
- $\#\{\text{extrema of } R\vec{\mu}\} \le \#\text{ extreme } \vec{\mu}$

$\Leftrightarrow$ well match property of entropy solution

Bad news: MPLR cannot be simple

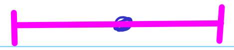
Lemma II.5.2.7. Linear monotonicity preserving reconstruction trivial

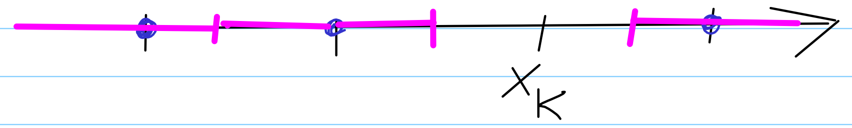
Every linear, monotonicity preserving ($\to$ Def. II.5.2.1) linear reconstruction yields piecewise constant functions.

(5) : R_m linear

$\Longleftrightarrow \quad R_m(\alpha\vec{\mu}+\beta\vec{\nu}) = \alpha R_m(\vec{\mu})+\beta R(\vec{\nu})$

Proof: $(\vec{\varepsilon}^k)_j = \begin{cases} 1, & j=k \\ 0, & j\neq k \end{cases} \triangleq$ a basis of $\mathbb{R}^{\mathbb{Z}}$

$R_m(\vec{e}^k) =$ 



$R_m(\vec{\mu}) = R_m(\sum \mu_j \vec{e}^j) = \sum \mu_j R_m(\vec{e}^k)$

▷ No higher order for linear MPLR

Definition ||.5.2.9. Minmod reconstruction

The minmod reconstruction R_{mm} is a piecewise linear reconstruction (→ Def. 8.5.1.3) defined by

$$(R_{mm}\vec{\mu})(x) = \mu_j + \sigma_j(x-x_j)$$
for $x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}, j\in\mathbb{Z}$,

$$\sigma_j := \operatorname{minmod}\left(\frac{\mu_{j+1}-\mu_j}{x_{j+1}-x_j}, \frac{\mu_j-\mu_{j-1}}{x_j-x_{j-1}}\right),$$

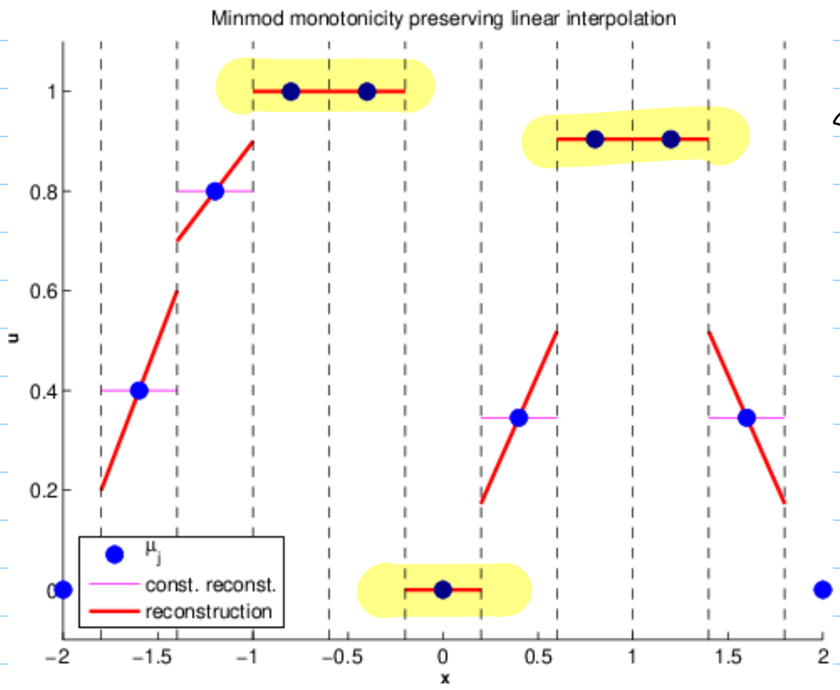
$$\operatorname{minmod}(v,w) := \begin{cases} v & , vw>0, |v|<|w|, \\ w & , vw>0, |w|<|v|, \\ 0 & , vw\leq 0 . \end{cases}$$

$\sigma_j = \sigma_j(\mu_{j-1},\mu_j,\mu_{j+1})$

local slopes

Pick smaller slope

Sign change of slopes



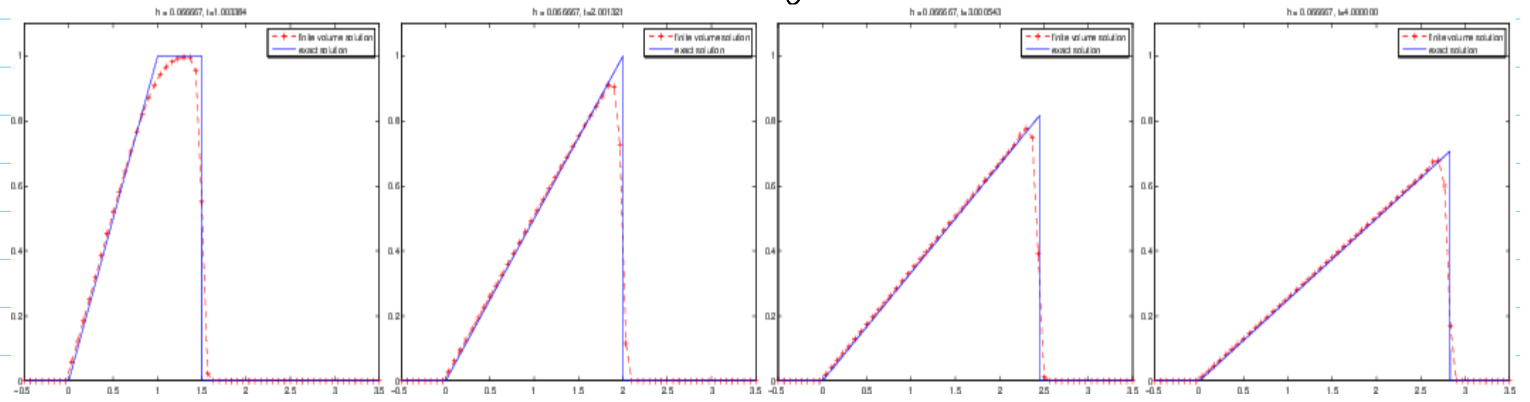
◁ Minmod MPLR of data

▷ Flat at extrema

Lemma ||.5.2.10. Monotonicity preservation of minmod reconstruction

Minmod reconstruction (→ Def. ||.5.2.9) is monotonicity preserving (→ Def. ||.5.2.1)

Example: • C.P. for Burgers equ., box initial data
• Godunov flux + minmod MPLR
• "exact timestepping"



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▷ Neither overshoots nor oscillations

Comparison with low-order FVM:

$\Rightarrow$ better resolution of rarefactions and shocks

For monotone smooth solutions

$\Rightarrow$ $O(h^2)$-cvg. in L^1-L^∞, L^∞-L^∞

11.5.3 MUSCL Scheme

(Monotone upwind scheme for conservation laws)

For smooth solutions

$$\|\text{space-time error}\| = O(\underbrace{h^p}_{\text{spatial order [cons. error]}} + \underbrace{\tau^q}_{\text{temporal order}})$$

CFL-condition: $\tau = O(h_m)$

By error balancing most efficient: $q = p$

▷ $q = 2$ for minimod-based 2nd-order FVM

↳ Explicit midpoint rule

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

Experiment: C.P. Burgers equ., box initial data

- Conservative finite volume spatial discretization (8.5.1.1) with monotone consistent 2-point flux, e.g., Godunov numerical flux (8.3.4.33)
- Piecewise linear reconstruction (→ Def. 8.5.1.3) with minmod slope limiting (→ Def. 8.5.2.9):

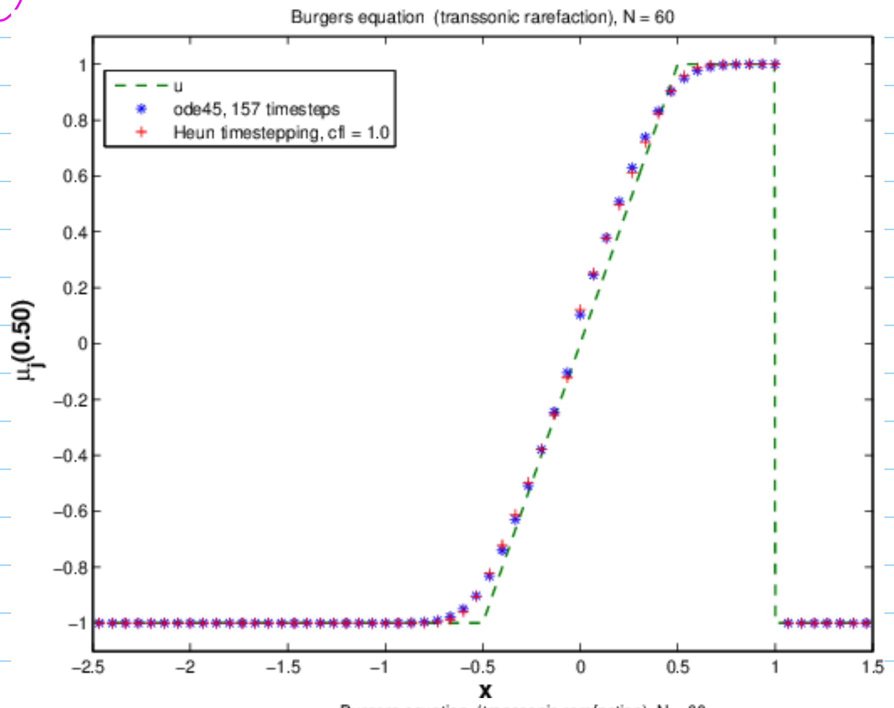
$$v_j^{\pm} := \mu_j \pm \tfrac{1}{2}\,\mathrm{minmod}\left(\mu_{j+1} - \mu_j, \mu_j - \mu_{j-1}\right). \qquad (8.5.3.1)$$

- 2nd-order Runge-Kutta timestepping for (8.5.1.1): method of Heun, *cf.* (8.4.1.11):

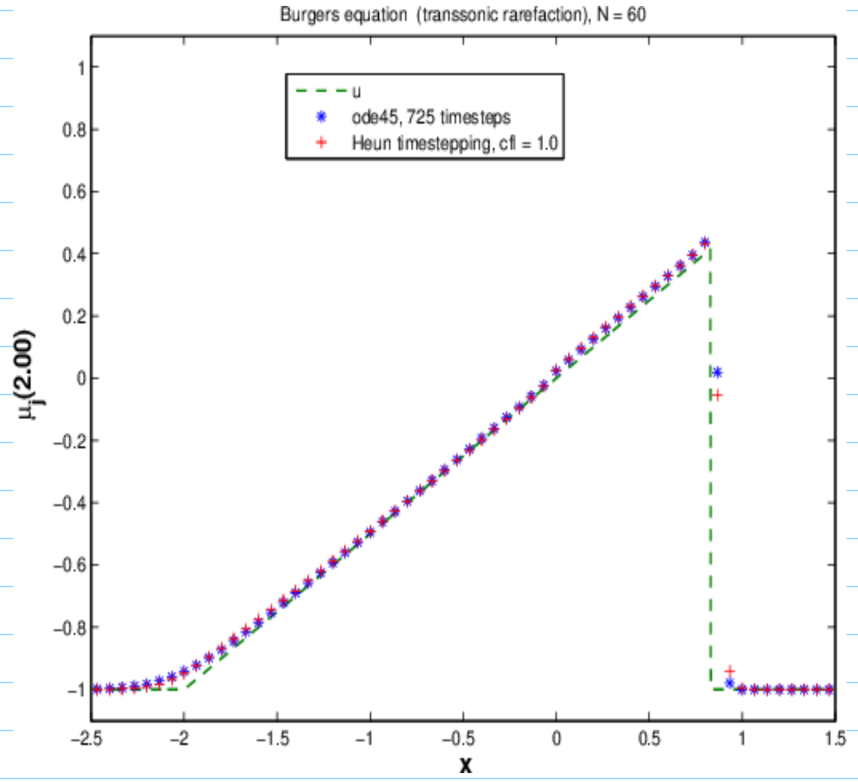
R.h.s of FV-MOL-ODE:

$$\begin{aligned} \mathcal{L}_h(\vec{\mu}) &:= -\frac{1}{h}\left(F(v_j^+(t), v_{j+1}^-(t)) - F(v_{j-1}^+(t), v_j^-(t))\right), \\ \vec{\kappa} &:= \vec{\mu}^{(k)} + \tfrac{1}{2}\tau\mathcal{L}_h(\vec{\mu}^{(k)}), \\ \vec{\mu}^{(k+1)} &:= \vec{\mu}^{(k)} + \tau h\mathcal{L}_h(\vec{\kappa}). \end{aligned} \qquad (8.5.3.2)$$

⑦



$t = 1/2$



$t = 2$

▷ 2nd-order time-stepping is sufficient ! ◁

Review question 11.5.3.5

[Answer without looking up information]

A:

Argue why a linear monotonicity preserving piecewise linear reconstruction must impose vanishing slopes throughout.

B:

Explain the meanings of "linear" in the phrase "non-*linear linear* reconstruction" (in the context of high-order finite volume methods for 1D conservation laws).

C:

Fully discrete evolutions for scalar conservation laws on uniform space-time meshes can be described by means of stencils.

What will be the width of the stencil for the fully discrete evolution operator, when using two-point numerical fluxes, piecewise linear reconstruction based on min-mod limiting, and a (generic) s-stage explicit RK-SSM for timestepping for the discretization of a scalar conservation law in 1D.