ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 11.6.2: 1D Non-Linear Systems of Conservation Laws

Prof. R. Hiptmair, SAM, ETH Zurich

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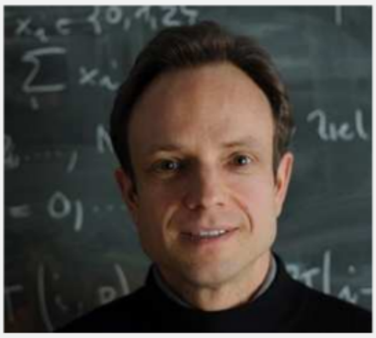


Prerequisites.

- Multi-dimensional differential calculus

Dependencies. [Lecture → Rem. 11.1.3.5], [Lecture → Section 11.2.3], [Lecture → Section 11.2.4], [Lecture → Section 11.2.5], [Lecture → Section 11.2.6].

Duration: 45 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XI. Numerical Methods for Conservation Laws

11.6. A Glimpse of Systems of Conservation Laws

11.6.2. 1D Non-Linear Systems of Conservation Laws

11.6.2.1. NLSCL: Definition and Fundamentals

Definition 11.6.2.2. Non-linear system of conservation laws

Given $m \in \mathbb{N}$, an open set $U \subset \mathbb{R}^m$, and a continuously differentiable **flux function** $\mathbf{F} : U \to \mathbb{R}^m$ a (translation invariant, “constant-coefficient”) **non-linear system of conservation laws** (NLSCL) in one spatial dimension for the unknown function $\mathbf{u} : D \subset \mathbb{R} \times \mathbb{R}^+ \to U$ reads

$$\frac{\partial \mathbf{u}}{\partial t}(x,t) + \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x}(x,t) = 0 \quad \text{on} \quad D\,. \tag{11.6.2.3}$$

Conservation:

$$\frac{d}{dt}\int_a^b \mathbf{u}(x,t)\,\mathrm{d}x = -(\mathbf{F}(u(b,t)) - \mathbf{F}(\mathbf{u}(a,t))) \quad \forall a,b \in \mathbb{R},\ t > 0\,. \tag{11.6.2.10}$$

§ 11.6.2.6: Properties of flux functions

Assumption 11.6.2.6. Strict hyperbolicity

The Jacobian $\mathsf{D}\mathbf{F}(\mathbf{u})$ of the flux function $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ must have m *distinct real eigenvalues* for any $\mathbf{u} \in U$.

Assumption 11.6.2.8. Genuine non-linearity/linear degeneracy

The eigenvectors $\mathbf{r}_i(\mathbf{u})$ and associated eigenvalues $\lambda_i(\mathbf{u})$ of $\mathbf{DF}(\mathbf{u})$ must satisfy

$$\mathbf{grad}_{\mathbf{u}}\,\lambda_i(\mathbf{u})\cdot\mathbf{r}_i(\mathbf{u}) \neq 0 \quad \forall \mathbf{u}\in U \quad \text{or} \quad \mathbf{grad}_{\mathbf{u}}\,\lambda_i(\mathbf{u})\cdot\mathbf{r}_i(\mathbf{u}) = 0 \quad \forall \mathbf{u}\in U\,. \tag{11.6.2.9}$$

$\lambda_i(\underline{u}) \triangleq$ ith eigenvalue of $DF(\underline{u})$, $r_i(\underline{u}) \triangleq$ EV

$m=1$: $f''(u) \neq 0 \to$ uniformly convex/concave

Definition 11.6.2.11. Weak solution of Cauchy problem for system of conservation laws

For $\mathbf{u}_0 \in (L^\infty(\mathbb{R}))^m$, $\mathbf{u}:\mathbb{R}\times]0,T[\mapsto\mathbb{R}^m$ is a **weak solution** of the Cauchy problem (11.6.2.4), if $\mathbf{u}\in(L^\infty(\mathbb{R}\times]0,T[))^m$ and

$$\int_{-\infty}^{\infty}\int_{0}^{T}\left\{\mathbf{u}(x,t)\cdot\frac{\partial\mathbf{\Phi}}{\partial t}(x,t)+\mathbf{F}(\mathbf{u}(x,t))\cdot\frac{\partial\mathbf{\Phi}}{\partial x}(x,t)\right\}\mathrm{d}t\mathrm{d}x-\int_{-\infty}^{\infty}\mathbf{u}_0(x)\cdot\mathbf{\Phi}(x,0)\,\mathrm{d}x=0\,,$$

for all $\mathbf{\Phi}\in(C_0^\infty(\mathbb{R}\times[0,T[))^m$ fulfilling $\mathbf{\Phi}(\cdot,T)=0$.

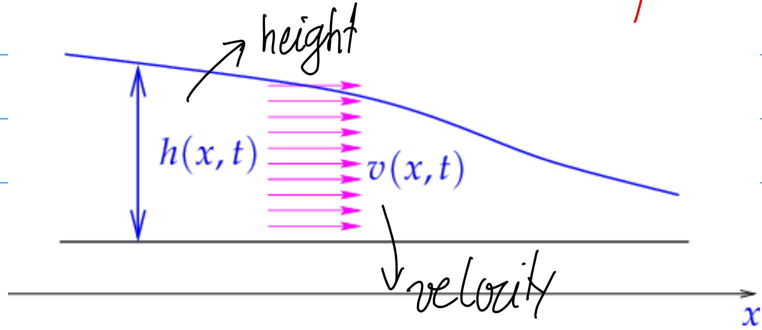
11.6.2.2. Examples

- 1D Euler equations, $m=3$

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho v\\ E\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\rho v\\ \rho v^2+p\\ (E+p)v\end{bmatrix}=0 \quad \triangleq \text{(11.6.2.3) with} \quad \mathbf{u}=\begin{bmatrix}\rho\\ \rho v\\ E\end{bmatrix},\quad \mathbf{F}(\mathbf{u})=\begin{bmatrix}u_2\\ u_2^2/u_1+p(\mathbf{u})\\ (u_3+p(\mathbf{u}))u_2/u_1\end{bmatrix},$$

$+\ p=p(\underline{u})$

- 1D shallow water equations, $m=2$



$\triangleleft$ uniform flow in a straight channel conservation of mass ▶ $$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(vh) = 0\,, \tag{11.6.2.14a}$$

conservation of momentum ▶ $$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(hv^2 + \tfrac{1}{2}gh^2) = 0\,. \tag{11.6.2.14b}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} h \\ hv \end{bmatrix}\,, \quad \mathbf{F}(\mathbf{u}) := \begin{bmatrix} u_2 \\ u_2^2/u_1 + \frac{1}{2}gu_1^2 \end{bmatrix} \blacktriangleright \; \mathsf{D}\mathbf{F}(\mathbf{u}) = \begin{bmatrix} 0 & 1 \\ -(u_2/u_1)^2 + gu_1 & 2u_2/u_1 \end{bmatrix}. \tag{11.6.2.15}$$

$\mathcal{U} := \mathbb{R}^+ \times \mathbb{R}$

11.6.2.3 Riemann Problem for NLSC

$$\frac{\partial \mathbf{u}}{\partial t}(x,t) + \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x}(x,t) = 0 \quad \text{on} \quad \mathbb{R}\times]0,T[\,, \quad \mathbf{u}(x,0) = \begin{cases} \mathbf{u}_l & \text{for} \quad x < 0\,, \\ \mathbf{u}_r & \text{for} \quad x > 0\,, \end{cases} \tag{11.6.2.17}$$

$\underline{u}_l, \underline{u}_r \in \mathbb{R}^m$

- Single shock solutions : Rankine-Hugoniot jump cond.

(RHJC) $\quad \dot{s}(\underline{u}_r - \underline{u}_l) = \underline{F}(\underline{u}_l) - \underline{F}(\underline{u}_r)$

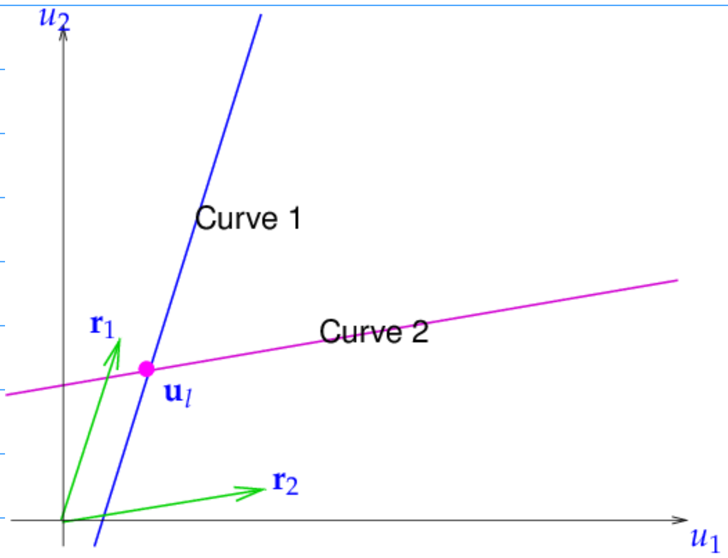
$\dot{s}$ ← constant shock speed

$F(\underline{u}) = A\underline{u} \Rightarrow \dot{s}$ eigenvalue of A

$\underline{u}_l \in \mathbb{R}^m$ fixed : (RHJC) $\hat{=}$ m equs. $\leftrightarrow$ $m+1$ unknowns

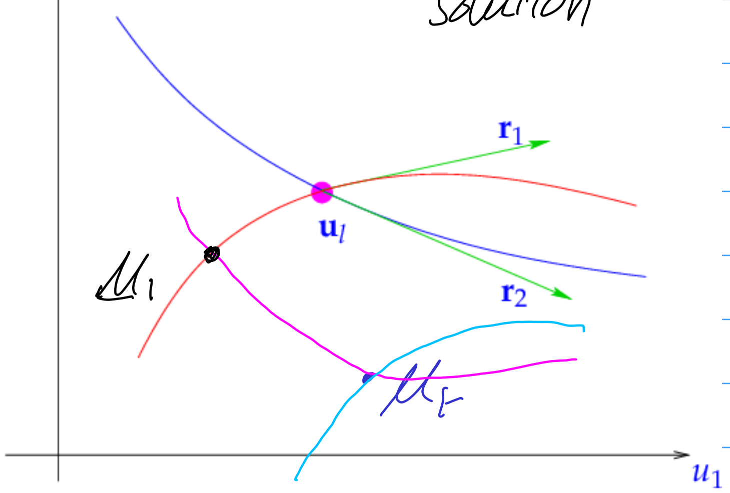
$\rightarrow$ 1-parameter families of solutions

$m = 2$: Visualization in state space



Linear case: $\mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u}$

$u_l, u_1, u_r \hat{=}$ shock-fan solution



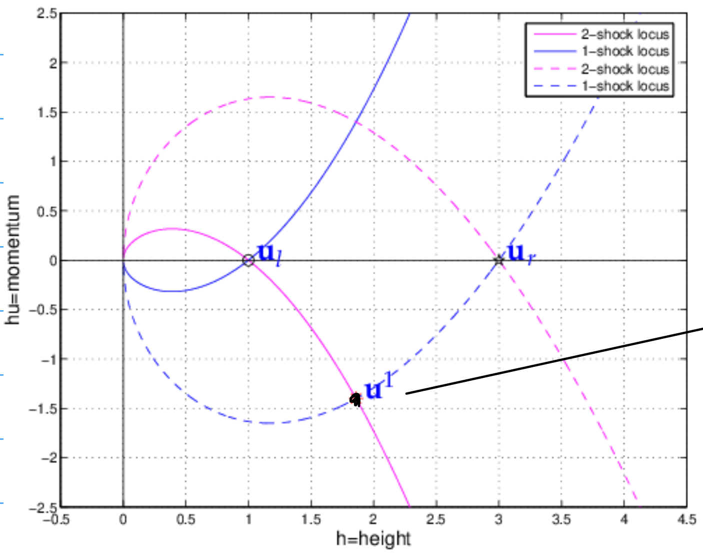
General non-linear case

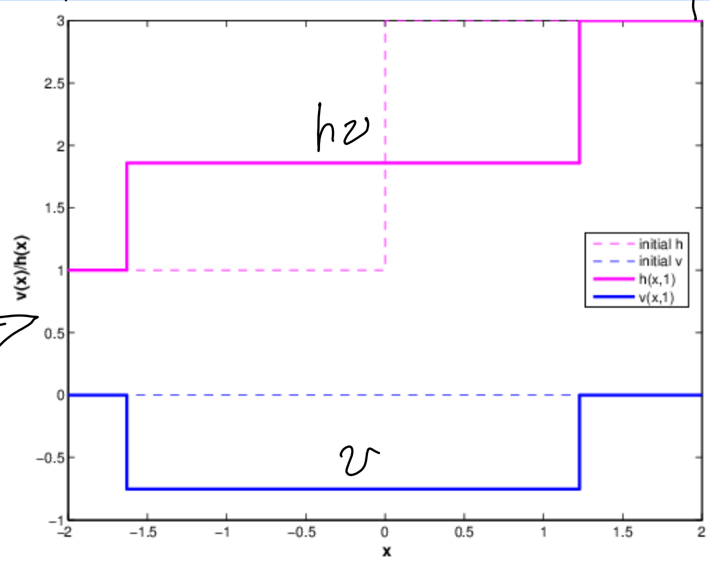
Hugoniot locus $\mathcal{H}(u_l)$ $\hat{=}$ u_r connected to u_l with a single shock sol.

Ex. 11.6.2.25 (Shock fans for SWE)

Dam break problem:

u_l u_r





shock fan

- Simple rarefaction fan :

$\hat{=}$ Continuous similarity solutions of R.P., $\underline{u}(x,t) = \underline{w}(x/t)$

$m=2$, weak solutions of R.P. :

2 shocks or shock + R.F. or 2. R.F.s

▷ Need entropy conditions to identify non-physical shocks

Definition 11.6.2.29. Lax entropy condition for systems of conservation laws

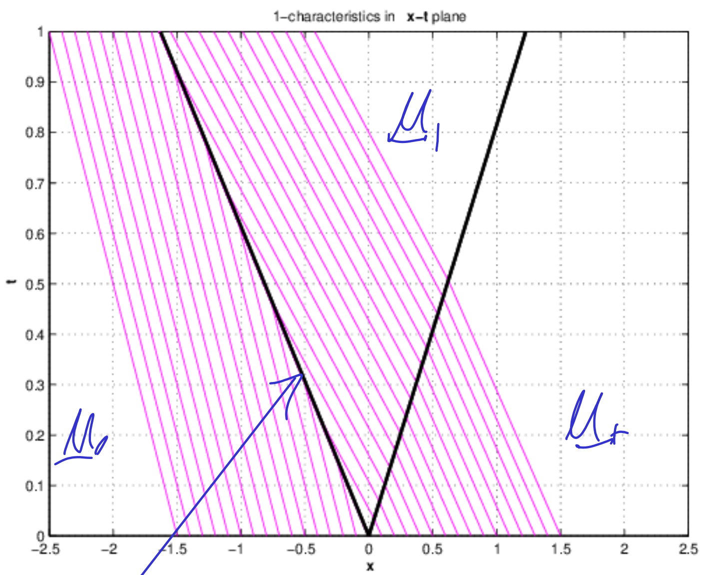
A discontinuity separating states $\mathbf{u}_l$ and $\mathbf{u}_r$ and propagating at speed $\dot{s}$ satisfies the Lax entropy condition (LEC), if

(i) $\exists k \in \{1,\dots,m\}$: $\lambda_k(\mathbf{u}_l) > \dot{s} > \lambda_k(\mathbf{u}_r)$
(ii) $\forall j < k$: $\lambda_j(\mathbf{u}_l), \lambda_j(\mathbf{u}_r) < \dot{s}$
(iii) $\forall j > k$: $\lambda_j(\mathbf{u}_l), \lambda_j(\mathbf{u}_r) > \dot{s}$

$\Leftrightarrow$ physical

Ex. 11.6.2.31 (LEC for SWE)

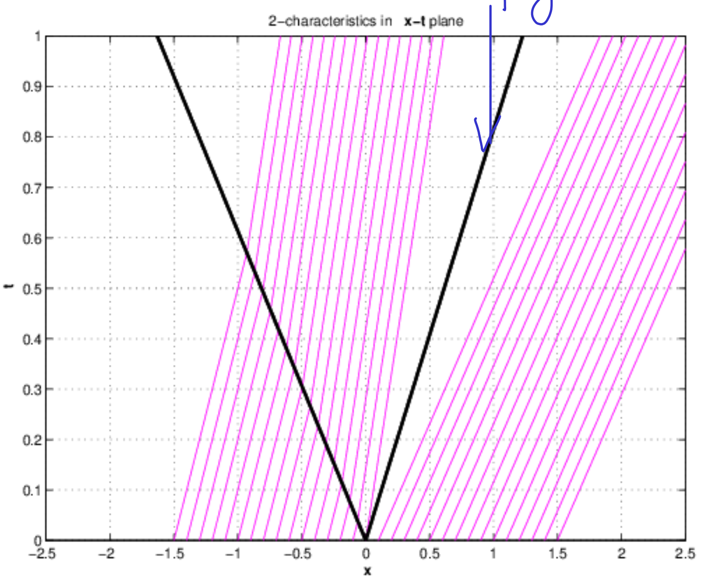
✦ Riemann problem for (11.6.2.14): $h_l = 1$, $h_r = 3$, $v_l = 0$, $v_r = 0$ (dam break), see Ex. 11.6.2.25, Fig. 614, Fig. 615



physical shock

1-characteristics

non-physical



2-characteristics
Lax entropy condition violated !

Solution of Riemann problem for non-linear systems

The computation of closed-form ("analytic") "physical" solutions of Riemann problems for non-linear systems of conservation laws is possible only for a few simple models. No general algorithm exists so far.

▷ Review questions 11.6.2.34