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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 11.6.3: Finite Volume Methods (FVMs) for Systems of Conservation Laws

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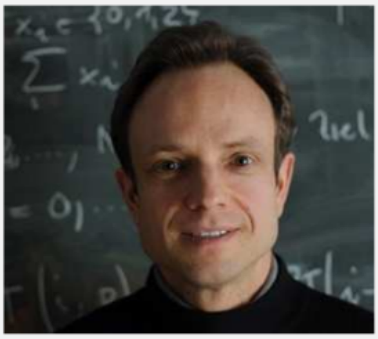


Prerequisites.

- Diagonalization of matrices
- Multi-dimensional differential calculus
- Curvature of functions

Dependencies. [Lecture → Section 11.3.2], [Lecture → Section 11.3.4], [Lecture → Section 11.6.1]

Duration: 65 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XI. Numerical Methods for Conservation Laws

11.6. A Glimpse of Systems of Conservation Laws

11.6.3. Finite Volume Methods for Systems of Conservation Laws in 1D

1D C.P.:

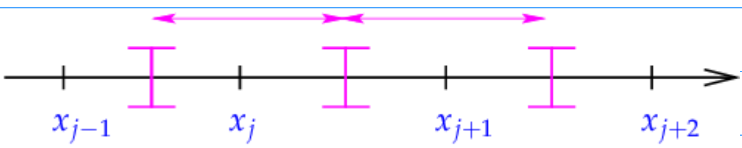
PDE (11.6.1.2): $$\frac{\partial \mathbf{u}}{\partial t}(x,t) + \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x}(x,t) = 0 \quad \text{on} \quad \mathbb{R}\times]0,T[\,, \qquad (11.6.2.4a)$$

Initial conditions: $$\mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{for} \quad x \in \mathbb{R}\,, \qquad (11.6.2.4b)$$

11.6.3.1 FV Spatial Semi-Discretization

$\mathcal{M} \hat{=}$ equidistant spatial mesh

$\mathcal{M} := \{]x_{j-1}, x_j[: x_j := jh,\ j \in \mathbb{Z}\}$. (11.3.2.1)



mesh cells and dual cells ▷

Unknowns: Dual cell averages

$$\boldsymbol{\mu}_j(t) \approx \frac{1}{h} \int_{x_{j-1/2}}^{j+1/2} \mathbf{u}(x,t)\,\mathrm{d}x \in \mathbb{R}^m\,, \quad t \in]0,T[\,. \qquad (11.6.3.3)$$

$$\frac{d\boldsymbol{\mu}_j}{dt}(t) = -\frac{1}{h}\left(\widehat{\mathbf{F}}(\boldsymbol{\mu}_{j-m_l+1}(t),\ldots,\boldsymbol{\mu}_{j+m_r}(t)) - \widehat{\mathbf{F}}(\boldsymbol{\mu}_{j-m_l}(t),\ldots,\boldsymbol{\mu}_{j+m_r-1}(t))\right),\quad j\in\mathbb{Z}, \tag{11.6.3.4}$$

+ truncation to bounded interval $\Rightarrow$ ODE

numerical flux $\widehat{F} : (\mathbb{R}^m)^{m_e+m_r} \to \mathbb{R}^m$

Most common : $m_r = m_e = 1$

11.6.3.2. Numerical fluxes for linear Systems of CL

PDE: $\dfrac{\partial \mathbf{u}}{\partial t}(x,t) + \mathbf{A}\dfrac{\partial \mathbf{u}}{\partial x}(x,t) = 0$ on $\mathbb{R}\times]0,T[$, (11.6.1.15a)

+ initial conditions: $\mathbf{u}(x,0) = \mathbf{u}_0(x)$ for $x\in\mathbb{R}$. (11.6.1.15b)

FVM with 2-point numerical flux

$$\frac{d\boldsymbol{\mu}_j}{dt}(t) = -\frac{1}{h}\left(\widehat{\mathbf{F}}(\boldsymbol{\mu}_j(t),\boldsymbol{\mu}_{j+1}(t)) - \widehat{\mathbf{F}}(\boldsymbol{\mu}_{j-1}(t),\boldsymbol{\mu}_j(t))\right) \tag{11.6.3.13}$$

FVM by diagonalization

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = 0 \quad \xrightarrow{\tilde{\mathbf{u}}:=\mathbf{R}^{-1}\mathbf{u}} \quad \underbrace{\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \mathbf{D}\frac{\partial \tilde{\mathbf{u}}}{\partial x} = 0}_{}, \quad \mathbf{D} := \operatorname{diag}(\lambda_1,\ldots,\lambda_m). \tag{11.6.3.9}$$

decoupled : $m\times$ scalar FVM applied componentwise

$\hookrightarrow$ numerical flux $\widetilde{F}$

$$\frac{d\tilde{\boldsymbol{\mu}}_j}{dt}(t) = -\frac{1}{h}\left(\widetilde{\mathbf{F}}(\tilde{\boldsymbol{\mu}}_j(t),\tilde{\boldsymbol{\mu}}_{j+1}(t)) - \widetilde{\mathbf{F}}(\tilde{\boldsymbol{\mu}}_{j-1}(t),\tilde{\boldsymbol{\mu}}_j(t))\right). \tag{11.6.3.11}$$

$\hookrightarrow$ Componentwise $\widehat{F}$ for $\frac{\partial u}{\partial t} + \lambda_i \frac{\partial u}{\partial x} = 0$

Trf. back:

$$\widehat{\mathbf{F}}(\mathbf{v},\mathbf{w}) := \mathbf{R}\widetilde{\mathbf{F}}(\mathbf{R}^{-1}\mathbf{v},\mathbf{R}^{-1}\mathbf{w}) \;, \quad \mathbf{v},\mathbf{w}\in\mathbb{R}^m \;, \tag{11.6.3.12}$$

- Upwind FVM ($\to$ Sect. 11.3.4.3)

For $\frac{\partial u}{\partial t} + \lambda_i \frac{\partial u}{\partial x} = 0$

$$F_{uw}(v,w) = \begin{cases} \lambda_i v & \text{, if } \lambda_i \geq 0 \;, \\ \lambda_i w & \text{, if } \lambda_i < 0 \;, \end{cases} = \max\{\lambda_i,0\}v + \min\{\lambda_i,0\}w \;, \quad v,w\in\mathbb{R} \;. \tag{11.6.3.15}$$

$$\widetilde{\mathbf{F}}_{uw}(\mathbf{v},\mathbf{w}) = \mathbf{D}^+\mathbf{v} + \mathbf{D}^-\mathbf{w} \;, \quad \begin{aligned} \mathbf{D}^+ &:= \operatorname{diag}(\max\{\lambda_i,0\},\, i=1,\dots,m) \;, \\ \mathbf{D}^- &:= \operatorname{diag}(\min\{\lambda_i,0\},\, i=1,\dots,m) \;. \end{aligned} \tag{11.6.3.16}$$

w/ (3.12):

$$\boxed{\widehat{\mathbf{F}}_{uw}(\mathbf{v},\mathbf{W}) = \mathbf{A}^+\mathbf{v} + \mathbf{A}^-\mathbf{w}} \;, \quad \mathbf{v},\mathbf{w}\in\mathbb{R}^m \;, \quad \mathbf{A}^{\pm} := \mathbf{R}\mathbf{D}^{\pm}\mathbf{R}^{-1} \;. \tag{11.6.3.17}$$

- Lax-Friedrich / Rusanov FVM ($\to$ Sect 11.3.4.2)

$$\frac{\partial \tilde{u}_i}{\partial t} + \lambda_i \frac{\partial \tilde{u}_i}{\partial x} = 0 \quad \blacktriangleright \quad F_{LF}(v,w) = \frac{1}{2}\lambda_i(v+w) - \frac{1}{2}|\lambda_i|(w-v) \;, \quad v,w\in\mathbb{R} \;. \tag{11.6.3.19}$$

$|\lambda| = \max(0,\lambda) - \min(0,\lambda)$

$$\widetilde{\mathbf{F}}_{LF}(\mathbf{v},\mathbf{w}) = \frac{1}{2}\mathbf{D}(\mathbf{v}+\mathbf{w}) - \frac{1}{2}|\mathbf{D}|(\mathbf{w}-\mathbf{v}) \;, \quad \mathbf{v},\mathbf{w}\in\mathbb{R}^m \;, \quad |\mathbf{D}| := \mathbf{D}^+ - \mathbf{D}^- \;. \tag{11.6.3.20}$$

w/ (3.12)

$$\boxed{\widehat{\mathbf{F}}_{LF}(\mathbf{v},\mathbf{w}) = \tfrac{1}{2}\mathbf{A}(\mathbf{v}+\mathbf{w}) - \tfrac{1}{2}|\mathbf{A}|(\mathbf{w}-\mathbf{v})} \;, \quad \mathbf{v},\mathbf{w}\in\mathbb{R}^m \;, \quad |\mathbf{A}| := \mathbf{A}^+ - \mathbf{A}^- \;. \tag{11.6.3.21}$$

· Godunov numerical flux

$\hat{F}_{GD}(\underline{v}, \underline{w}) = \underline{F}(\underline{u}(0,t))$

similarity solution of R.P.

$$\frac{\partial \mathbf{u}}{\partial t}(x,t) + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x}(x,t) = 0 \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^+ , \quad \mathbf{u}(x,0) = \begin{cases} \mathbf{v} & \text{for} \quad x < 0 , \\ \mathbf{w} & \text{for} \quad x \geq 0 . \end{cases} \tag{11.6.1.29}$$

(1.31) : shock fan solution :

$$\overline{\mathbf{u}}(x,t) = \begin{cases} \mathbf{v} & \text{for} \quad x < \lambda_1 t , \\ w_1^r \mathbf{r}_1 + \sum_{i=2}^{m} w_1^l \mathbf{r}_i & \text{for} \quad \lambda_1 t < x < \lambda_2 t , \\ \vdots & \vdots \\ \sum_{i=1}^{k} w_i^r \mathbf{r}_i + \sum_{i=k+1}^{m} w_1^l \mathbf{r}_i & \text{for} \quad \lambda_k t < x < \lambda_{k+1} t , \\ & k = 1,\ldots,m-1 , \\ \vdots & \vdots \\ \mathbf{w} & \text{for} \quad x > \lambda_m t , \end{cases} \tag{11.6.1.31}$$

$$\mathbf{u}_l = \sum_{k=1}^{m} w_k^l \mathbf{r}_k , \quad w_k^l := \left(\mathbf{R}^{-1}\mathbf{v}\right)_k \quad , \quad \mathbf{u}_r = \sum_{k=1}^{m} w_k^r \mathbf{r}_k , \quad w_k^r := \left(\mathbf{R}^{-1}\mathbf{w}\right)_k . \tag{11.6.1.30}$$

$$\overline{\mathbf{u}}^{\downarrow}(\mathbf{v},\mathbf{w}) := \overline{\mathbf{u}}(0,t) = \sum_{i=1}^{p} w_i^r \mathbf{r}_i + \sum_{i=p+1}^{m} w_i^l \mathbf{r}_i , \quad p := \max\{i \in \{1,\ldots,m\} : \lambda_i < 0\} . \tag{11.6.3.23}$$

$$\mathbf{A}\overline{\mathbf{u}}^{\downarrow}(\mathbf{v},\mathbf{w}) = \sum_{i=1}^{p} w_i^r \lambda_i \mathbf{r}_i + \sum_{i=p+1}^{m} w_i^l \lambda_i \mathbf{r}_i = \sum_{i=1}^{p} \left(\mathbf{R}^{-1}\mathbf{w}\right)_i \lambda_i \mathbf{r}_i + \sum_{i=p+1}^{m} \left(\mathbf{R}^{-1}\mathbf{v}\right)_i \lambda_i \mathbf{r}_i$$
$$= \mathbf{R}\mathbf{D}^{-}\mathbf{R}^{-1}\mathbf{w} + \mathbf{R}\mathbf{D}^{+}\mathbf{R}^{-1}\mathbf{v} = \mathbf{A}^{+}\mathbf{v} + \mathbf{A}^{-}\mathbf{w} ,$$

$\triangleright \quad \hat{F}_{GD} = \hat{F}_{UW}$

11.6.3.3. Numerical Fluxes for NLSCLs

Recall: Godunov numerical flux

$$\boxed{\widehat{\mathbf{F}}_{GD}(\mathbf{v},\mathbf{w}) := \mathbf{F}(\mathbf{u}^{\downarrow}(\mathbf{v},\mathbf{w}))}\;, \quad \mathbf{v},\mathbf{w} \in U \subset \mathbb{R}^m\,, \quad \mathbf{u}^{\downarrow}(\mathbf{v},\mathbf{w}) := \mathbf{\Psi}(0)\,, \qquad (11.6.3.26)$$

$\bar{u}(x,t) = \Psi(x/t)$

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \frac{\partial \mathbf{F}(\overline{\mathbf{u}})}{\partial x} = 0 \quad , \quad \overline{\mathbf{u}}(x,t) = \begin{cases} \mathbf{v} & \text{for } x < 0\,, \\ \mathbf{w} & \text{for } x \geq 0\,. \end{cases} \qquad (11.6.3.27)$$

Rem.: Exact Godunov FVM not feasible/ not affordable in general

▷ Use approximate Godunov N.F.

Replace (3.27) w/ a "linear R.P."

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t}(x,t) + \mathbf{A}_R \frac{\partial \tilde{\mathbf{u}}}{\partial x}(x,t) = 0 \;\text{ on }\; \mathbb{R}\times\mathbb{R}^+ \quad , \quad \tilde{\mathbf{u}}(x,t) = \begin{cases} \mathbf{v} & \text{for } x < 0\,, \\ \mathbf{w} & \text{for } x \geq 0\,, \end{cases} \qquad (11.6.3.29)$$

↳ $A_R = A_R(\underline{v},\underline{w})$

"Correct shock speed preservation"

If $\underline{w} \in \mathcal{H}(\underline{v}) \Rightarrow$ single shock solution, moving with velocity $\dot{s}$

(RHJC) $\Rightarrow$ $\dot{s}(\underline{w}-\underline{v}) = F(\underline{w}) - F(\underline{v})$

We demand: Same single-shock solution for (3.29)

$$A_R(\underline{w}-\underline{v}) = \dot{s}(\underline{w}-\underline{v}) = F(\underline{w}) - F(\underline{v})$$

Definition 11.6.3.30. Roe matrix

A matrix-value function $\mathbf{A}_R : U \times U \to \mathbb{R}^{m,m}$ is a **Roe matrix** for the flux function $\mathbf{F} : U \to \mathbb{R}^m$, if

(i) $\mathbf{A}_R(\mathbf{u},\mathbf{u}) = \mathsf{D}\mathbf{F}(\mathbf{u})$ for all $\mathbf{u} \in U$ (consistency),

(ii) $\mathbf{A}_R(\mathbf{v},\mathbf{w})(\mathbf{w}-\mathbf{v}) = \mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v})$ for all $\mathbf{v},\mathbf{w} \in U$ (correct shock speed),

(iii) $\mathbf{A}_R(\mathbf{v},\mathbf{w})$ has m distinct real eigenvalues (strict hyperbolicity).

Finding A_R is an art!

Ex. 11.6.3.31 (Roe matrix SWE)

conservation of mass ▶ $$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(vh) = 0\,, \qquad (11.6.2.14a)$$

conservation of momentum ▶ $$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(hv^2 + \tfrac{1}{2}gh^2) = 0\,. \qquad (11.6.2.14b)$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} h \\ hv \end{bmatrix}, \quad \mathbf{F}(\mathbf{u}) := \begin{bmatrix} u_2 \\ u_2^2/u_1 + \frac{1}{2}gu_1^2 \end{bmatrix} \;\blacktriangleright\; \mathsf{D}\mathbf{F}(\mathbf{u}) = \begin{bmatrix} 0 & 1 \\ -(u_2/u_1)^2 + gu_1 & 2u_2/u_1 \end{bmatrix}. \qquad (11.6.2.15)$$

$\vdots$

$$\mathbf{A}_R(\mathbf{v},\mathbf{w}) = \mathsf{D}\mathbf{F}\left(\begin{bmatrix} \overline{h} \\ \overline{h}\widehat{v} \end{bmatrix}\right), \quad \begin{aligned} \overline{h} &:= \tfrac{1}{2}(h_v + h_w)\,, \\ \widehat{v} &:= \frac{\sqrt{h_v}v_v + \sqrt{h_w}v_w}{\sqrt{h_v} + \sqrt{h_w}}\,, \end{aligned} \quad \mathbf{v} = \begin{bmatrix} h_v \\ h_v v_v \end{bmatrix}, \; \mathbf{w} = \begin{bmatrix} h_w \\ h_w v_w \end{bmatrix}. \qquad (11.6.3.32)$$

Roe average

Problem: $m=1$, Ex. 11.3.4.22

→ Non-physical shock solution of (3.29)

§ 11.6.3.33 (HLLE "entropy fix")

Idea: Insert an auxiliary "rarefaction" state.

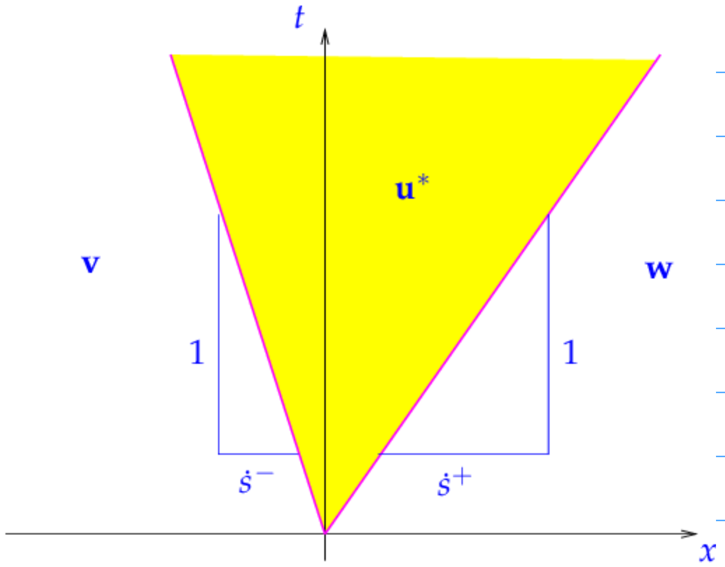
shock fan w/ 3 waves

$$\breve{\mathbf{u}}(x,t) = \begin{cases} \mathbf{v} & \text{, if } x < \dot{s}^- t\,, \\ \mathbf{u}^* & \text{, if } \dot{s}^- t \leq x < \dot{s}^+ t\,, \\ \mathbf{w} & \text{, if } \dot{s}^+ t \leq x\,, \end{cases} \qquad (11.6.3.34)$$

with edge speeds

$$\begin{aligned} \dot{s}^- &:= \min\{\widehat{\lambda}_1, \lambda_1(\mathbf{v})\}\,, \\ \dot{s}^+ &:= \max\{\widehat{\lambda}_m, \lambda_m(\mathbf{w})\}\,, \end{aligned} \qquad (11.6.3.35)$$

where the $\widehat{\lambda}_i$, $i = 1, \dots, m$, are the (sorted) eigenvalues of a Roe matrix $\mathbf{A}_R(\mathbf{v}, \mathbf{w})$.



Given $\dot{s}^\pm$ the "bridge state" u^* is obtain from global conservation

$$-\frac{d}{dt}\int_{\mathbb{R}} \breve{u}(x,t)\,\mathrm{d}x = \mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v})\,. \qquad (11.6.3.36)$$

$$\dot{s}^-(\mathbf{u}^* - \mathbf{v}) + \dot{s}^+(\mathbf{w} - \mathbf{u}^*) = \mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v}) \quad \Rightarrow \quad \mathbf{u}^* = \frac{\mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v}) - \dot{s}^+\mathbf{w} + \dot{s}^-\mathbf{v}}{\dot{s}^- - \dot{s}^+}\,, \qquad (11.6.3.37)$$

$$\boxed{\widehat{\mathbf{F}}_{\mathrm{HLLE}}(\mathbf{v}, \mathbf{w}) = \begin{cases} \mathbf{F}(\mathbf{v}) & \text{, if } \dot{s}^- > 0 & \text{, } \dot{s}^- \text{ from (11.6.3.35)}\,, \\ \mathbf{F}(\mathbf{u}^*) & \text{, if } \dot{s}^- < 0 < \dot{s}^+ & \text{, } \mathbf{u}^* \text{ as in (11.6.3.37)}\,, \\ \mathbf{F}(\mathbf{w}) & \text{, if } \dot{s}^+ < 0 & \text{, } \dot{s}^+ \text{ according to (11.6.3.35)}\,, \end{cases}} \qquad (11.6.3.38)$$

$v, w \in \mathcal{U}$

Ex. 11.6.3.39 (HLLE numerical flux, m=1)

$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$, $f : \mathbb{R} \to \mathbb{R}$ C^1 & strictly convex

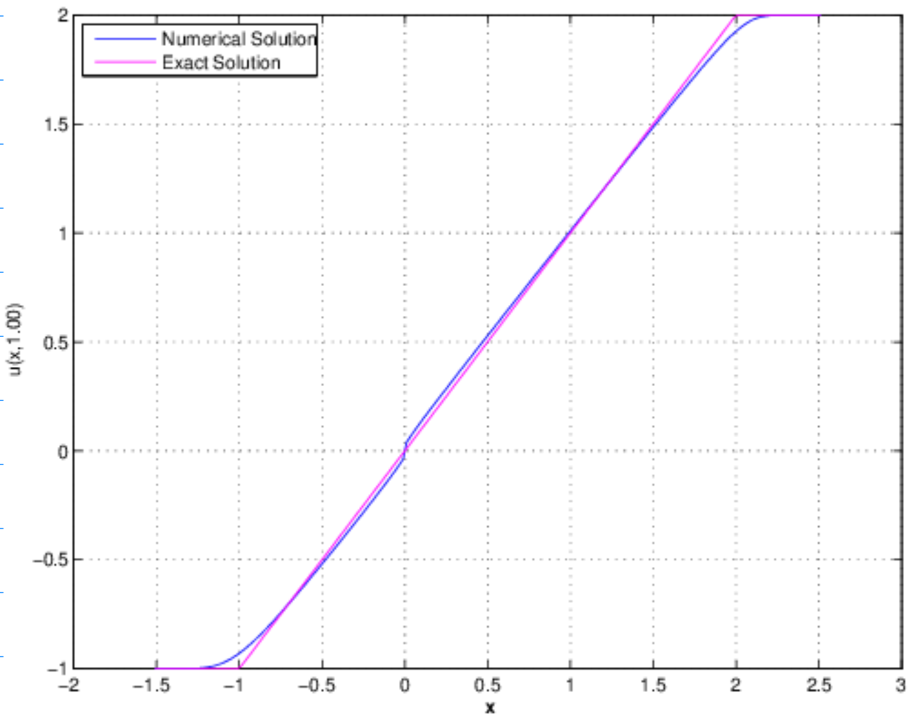
$\triangleright$ $A_R(v,w) = [\dot{s}]$, $\dot{s} = \begin{cases} f'(v), & v = w \\ \frac{f(w) - f(v)}{w - v}, & v \neq w \end{cases}$

$\dot{s}^- = \begin{cases} \dot{s}, & v > w \\ f'(v), & v \leq w \end{cases}$, $\dot{s}^+ = \begin{cases} \dot{s}, & v > w \\ f'(w), & v \leq w \end{cases}$

$$F_{\mathrm{HLLE}}(v,w) = f(\breve{u}^{\downarrow}) \,, \quad \breve{u}^{\downarrow} := \begin{cases} v & \text{, if } \dot{s}^- > 0 \,, \\ \frac{f(w)-f(v)-f'(w)w+f'(v)v}{f'(v)-f'(w)} & \text{, if } f'(v) < 0 < f'(w) \,, \\ w & \text{, if } \dot{s}^+ < 0 \,. \end{cases} \tag{11.6.3.42}$$

Case of transonic rarefaction

Exp. 11.6.3.43 (HLLE-FVM for Burgers equ.)



$\triangle$ Transonic rarefaction resolved well

Exp. 11.6.3.46 (HLLE-FVM for SWE)

Dam break problem, explicit Euler t.s.

§ 11.6.3.46 (Lax-Friedrich / Rusanov N.F.)

Linear case $F(\underline{u}) = A \cdot \underline{u}$

$$\boxed{\widehat{\mathbf{F}}_{\mathrm{LF}}(\mathbf{v},\mathbf{w}) = \underbrace{\tfrac{1}{2}\mathbf{A}(\mathbf{v}+\mathbf{w})}_{\text{central flux}} - \underbrace{\tfrac{1}{2}|\mathbf{A}|(\mathbf{w}-\mathbf{v})}_{\text{diffusive flux}}}\ , \quad \mathbf{v},\mathbf{w}\in\mathbb{R}^m\ , \quad |\mathbf{A}| := \mathbf{A}^+ - \mathbf{A}^-\ . \tag{11.6.3.21}$$

central flux

→ $\frac{1}{2}(F(\underline{v}) + F(\underline{w}))$

diffusive flux

⇓

Idea: Replace $|A| \to |A_R|$

$$\boxed{\widehat{\mathbf{F}}_{\mathrm{LF}}(\mathbf{v},\mathbf{w}) = \frac{1}{2}(\mathbf{F}(\mathbf{v})+\mathbf{F}(\mathbf{w})) - \frac{1}{2}|\mathbf{A}_R(\mathbf{v},\mathbf{w})|(\mathbf{w}-\mathbf{v})}\ , \quad \mathbf{v},\mathbf{w}\in U\ , \tag{11.6.3.48}$$

$$|\mathbf{A}_R(\mathbf{v},\mathbf{w})| := \mathbf{R}\,\mathrm{diag}(|\widehat{\lambda}_1|,\dots,|\widehat{\lambda}_m|)\mathbf{R}^{-1}\ , \quad \mathbf{A}_R(\mathbf{v},\mathbf{w}) = \mathbf{R}\,\mathrm{diag}(\widehat{\lambda}_1,\dots,\widehat{\lambda}_m)\mathbf{R}^{-1}\ .$$

Exp. 11.6.3.49 (L.F.-FVM for SWE)

→ dam break problem

Rem. 11.6.3.6: Timestepping for FVM semi-discretization as in Sect. 11.4.

▷ Review questions 11.6.3.50