12.1 Video Unit for [Lecture $\rightarrow$ Section 12.1]

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 12.1: Modeling the Flow of a Viscous Fluid

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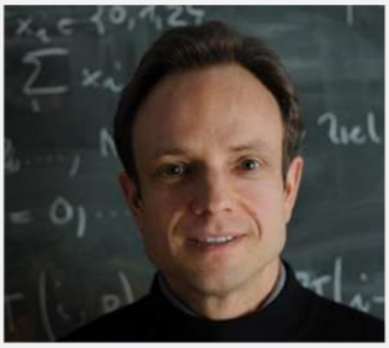


Prerequisites.

- Vector analysis, in particular the operators **curl** and div

Dependencies. Section 10.1,

Duration: 30 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XII. Finite Elements for the Stokes Equations

CFD: Simulation of the stationary flow of a viscous, incompressible in containing $\Omega \subset \mathbb{R}^d$, $d=2,3$

→ Compute approximate velocity field $\underline{v}: \Omega \to \mathbb{R}^d$

12.1. Modeling the Flow of a Viscous Fluid

Thermodynamic perspective: $\operatorname{div} \underline{v} = 0$ in Ω

§ 12.1.0.2 Creeping flows

Reynolds number $\quad \mathrm{Re} := \dfrac{\rho V L}{\mu}\,,$ (12.1.0.3)

where (for $d=3$)
- $\rho \mathrel{\hat=}$ density of the fluid ($[\rho] = \mathrm{kg\,m^{-3}}$)
- $V \mathrel{\hat=}$ typical velocity of the fluid ($[V] = \mathrm{m\,s^{-1}}$)
- $L \mathrel{\hat=}$ characteristic length of the region of interest Ω ($[L] = \mathrm{m}$)
- $\mu \mathrel{\hat=}$ dynamic viscosity ($[\mu] = \mathrm{kg\,m^{-1}s^{-1}}$)

This section: $\mathrm{Re} \ll 1$

[slow flows, small Ω, sticky]

§ 12.1.0.4 (Boundary conditions)

no-slip b.c. : $\underline{v} = 0$ on $\partial\Omega$

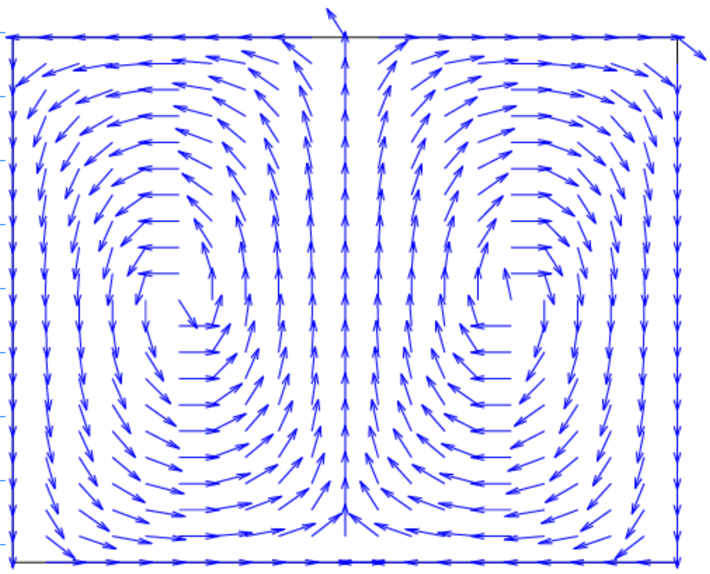
▷ Velocity configuration space

$$V := \left\{ \mathbf{v} : \overline{\Omega} \mapsto \mathbb{R}^d \text{ continuous, } \operatorname{div} \mathbf{v} = 0, \; \mathbf{v}_{|\partial\Omega} = 0 \right\}. \tag{12.1.0.6}$$

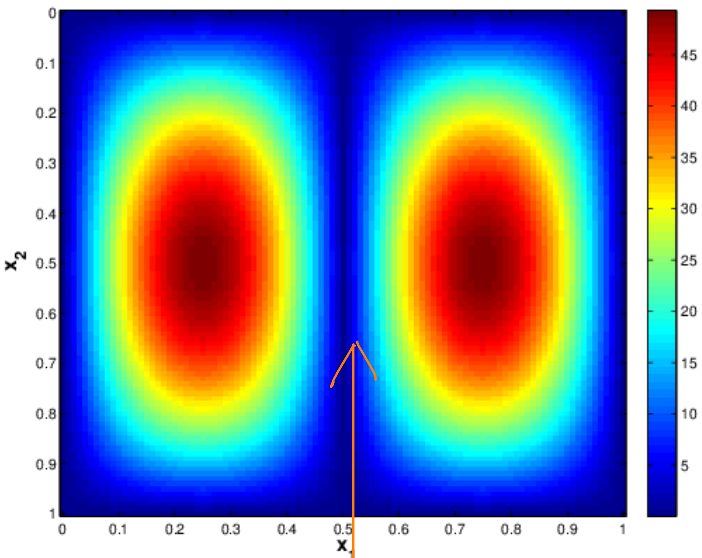
Assume:]We take for granted that the energy dissipation for a viscous fluid moving with velocity $\mathbf{v} : \Omega \to \mathbb{R}^d$

$$P_{\text{diss}}(\mathbf{v}) = \int_\Omega \mu \|\mathbf{curl}\, \mathbf{v}(\boldsymbol{x})\|^2 \, \mathrm{d}\boldsymbol{x} \tag{12.1.0.8}$$

rotation operator : measures strength of eddies



“eddy field”, $\operatorname{div} \mathbf{v} = 0$



Plot of $\|\mathbf{curl}\, \mathbf{v}\|$ for eddy field

laminar flow

(8) ⇒ Conversion : kinetic energy → heat mainly occurs in eddies

Extremal principle:

2nd law of T.D.: $P_{diss} \longrightarrow \max$

1st law of T.D.: Conservation of energy

$f: \Omega \to \mathbb{R}^3 \hat{=}$ stirring force field

$$\int_\Omega \mu \|\mathbf{curl}\,\mathbf{v}(x)\|^2\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x\,. \qquad (12.1.0.12)$$

dissipated energy energy injected through forces

▷ Equilibrium condition

$$\mathbf{v}^* = \operatorname{argmax}\left\{\int_\Omega \mu\|\mathbf{curl}\,\mathbf{v}(x)\|^2\,\mathrm{d}x\colon \mathbf{v}\in V\,,\ \mathbf{v} \text{ satisfies (12.1.0.12)}\right\}\,. \qquad (12.1.0.13)$$

§ 12.1.0.14 Dual equilibrium condition

Discussion in $\mathbb{R}^n$

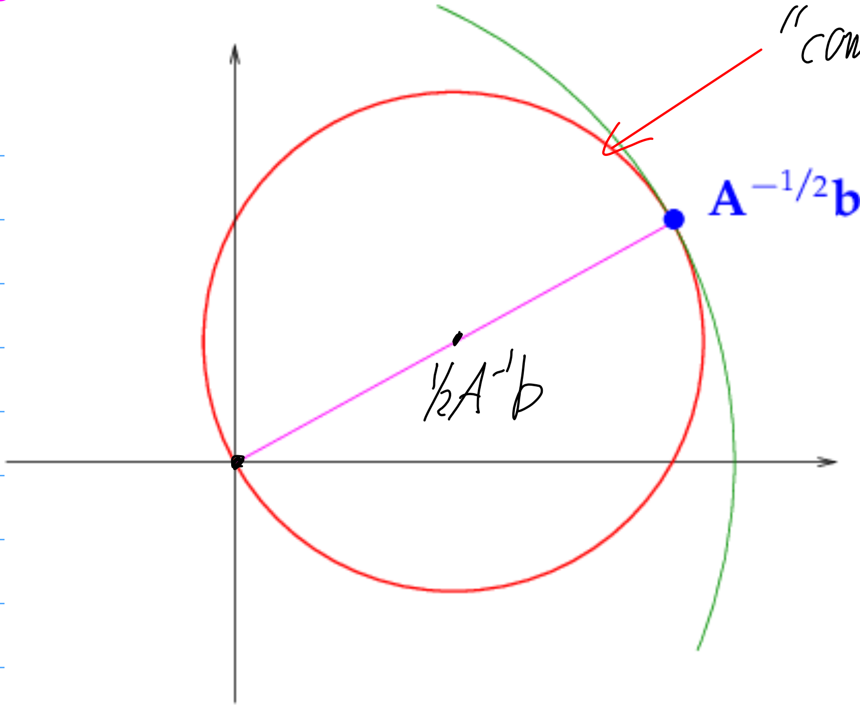
$$\mathbf{x}^* = \operatorname*{argmax}_{\mathbf{x}\colon \mathbf{x}^T\mathbf{A}\mathbf{x}=\mathbf{b}^T\mathbf{x}} \mathbf{x}^T\mathbf{A}\mathbf{x}\,, \qquad (12.1.0.15)$$

$A \in \mathbb{R}^{n,n}$ s.p.d

$\longleftrightarrow$ (12)

$\underline{y} := A^{1/2}\underline{x}$

$$\mathbf{x}^* = \operatorname*{argmax}_{(\mathbf{A}^{1/2}\mathbf{x})^\top(\mathbf{A}^{1/2}\mathbf{x})=(\mathbf{A}^{-1/2}\mathbf{b})^\top(\mathbf{A}^{1/2}\mathbf{x})} (\mathbf{A}^{1/2}\mathbf{x})^\top(\mathbf{A}^{1/2}\mathbf{x}) = \mathbf{A}^{-1/2}\operatorname*{argmax}_{\underbrace{\mathbf{y}\colon \|\mathbf{y}\|^2=(\mathbf{A}^{-1/2}\mathbf{b})^T\mathbf{y}}_{\text{"sphere", center } \frac{1}{2}A^{-1/2}\underline{b}}} \|\mathbf{y}\|^2\,. \qquad (12.1.0.16)$$



$= y$

$\Rightarrow \underline{x} = A^{-1}\underline{b}$

Sect. 1.2.3 :

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad \Leftrightarrow \quad \mathbf{x}^* = \underset{\mathbf{x}\in\mathbb{R}^n}{\operatorname{argmin}} \tfrac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{x} \,. \tag{12.1.0.17}$$

▷ 2nd equilibrium condition

$$\mathbf{v}^* = \underset{\mathbf{v}\in V}{\operatorname{argmin}} \tfrac{1}{2}\int_\Omega \mu \|\mathbf{curl}\,\mathbf{v}(x)\|^2 \,\mathrm{d}x - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x \quad . \tag{12.1.0.18}$$

$\hat{=}$ quadratic minim. prb.