ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 12.2.1: The Stokes Equations: Constrained Variational Formulation

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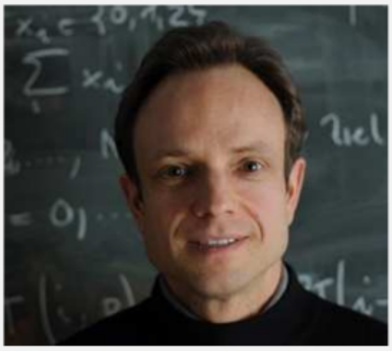

Prerequisites.

- Elementary mulit-dimensional calculus,

Dependencies. [Lecture → Section 1.2], [Lecture → Section 1.3], [Lecture → Section 1.4], [Lecture → Section 12.1]

Duration: 20 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XII. Finite Elements for the Stokes Equations

12.2. The Stokes Equation

12.2.1. Constrained Variational Formulation

QMP:

$$\mathbf{v}^* = \underset{\mathbf{v}\in V}{\operatorname{argmin}} \tfrac{1}{2}\int_\Omega \mu\|\mathbf{curl}\,\mathbf{v}(\boldsymbol{x})\|^2\,\mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\boldsymbol{x}\ . \qquad (12.1.0.18)$$

$$V := \left\{\mathbf{v} : \overline{\Omega} \mapsto \mathbb{R}^d \text{ continuous},\ \operatorname{div}\mathbf{v} = 0,\ \mathbf{v}_{|\partial\Omega} = 0\right\}. \qquad (12.1.0.6)$$

Lemma 12.2.1.2. $-\Delta = \mathbf{curl}\,\mathbf{curl} - \mathbf{grad}\,\operatorname{div}$

For $\mathbf{v} \in C^2(\overline{\Omega})$, $\mathbf{v}_{|\partial\Omega} = 0$, *holds* ($\mathbf{v} = [v_1, \dots, v_d]^\top$)

Jacobian

$$\int_\Omega \|\mathbf{curl}\,\mathbf{v}\|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega |\operatorname{div}\mathbf{v}|^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega \|\mathsf{D}\mathbf{v}\|_F^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathsf{D}\mathbf{v} : \mathsf{D}\mathbf{v}\,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^d \int_\Omega \|\mathbf{grad}\,v_i\|^2\,d\boldsymbol{x}\ .$$

Euclidean inner product of mats.

(12.1.0.18) $\overset{\text{Lemma 12.2.1.2}}{\Longleftrightarrow}$
$$\mathbf{v}^* = \underset{\mathbf{v}\in V}{\operatorname{argmin}} \tfrac{1}{2}\underbrace{\int_\Omega \mu\|\mathsf{D}\mathbf{v}\|_F^2\,\mathrm{d}\boldsymbol{x}}_{=:\mathsf{a}(\mathbf{v},\mathbf{v})} - \underbrace{\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\boldsymbol{x}}_{=:\ell(\mathbf{v})}\ . \qquad (12.2.1.4)$$

$$a(\underline{v},\underline{v}) = \sum_{i=1}^d \|\mathrm{grad}\,v_i\|^2_{L^2(\Omega)}$$

$$a(\underline{v},\underline{v}) = 0 \Rightarrow \mathrm{grad}\,v_i = 0 \Rightarrow v_i = \text{const} \Rightarrow v_i = 0$$

$$\underline{v}_{|\partial\Omega} = 0$$

$\triangleright$ a s.p.d.

Is (1.4) decoupled? No! $\operatorname{div} \mathbf{v} = 0$ couples components!

$$\text{(12.2.1.4)} \quad \Leftrightarrow \quad \underset{\mathbf{v}\in V}{\operatorname{argmin}} \sum_{i=1}^{d} \left(\tfrac{1}{2} \int_\Omega \mu \|\mathbf{grad}\, v_i\|^2 \,\mathrm{d}\mathbf{x} - \int_\Omega f_i v_i \,\mathrm{d}\mathbf{x} \right). \qquad \text{(12.2.1.10)}$$

§ 12.2.1.12 Function space framework

$\hat{=} \; \{ \mathbf{v} \in V : \; \|\mathbf{v}\|_a < \infty \}$

energy norm induced by $a(\cdot,\cdot)$

$\|\mathbf{v}\|_a^2 = \sum_{i=1}^{d} \|\mathbf{grad}\, v_i\|_{L^2(\Omega)} < \infty \iff v_i \in H^1(\Omega)$

$$H_0^1(\operatorname{div} 0, \Omega) := \left\{ \mathbf{v} \in (H_0^1(\Omega))^d : \operatorname{div} \mathbf{v} = 0 \right\} \qquad \text{(12.2.1.13)}$$

Sect. 1.4 : QMP $\iff$ LVP

$\mathbf{v} \in H_0^1(\operatorname{div} 0, \Omega)$: $\quad a(\mathbf{v}, \mathbf{w}) = \ell(\mathbf{w}) \quad \forall \mathbf{w} \in \ldots$

P.F. inequ.: $\|\cdot\|_a \approx \|\cdot\|_{H^1(\Omega)}$ on $H_0^1(\operatorname{div} 0, \Omega)$

$\Rightarrow$ Existence & uniqueness of sol. of (1.4)

BAD NEWS: No simple FE subspace of $H_0^1(\operatorname{div} 0, \Omega)$!

[2D: minimal polynomial degree = 5]