ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 12.2.2: The Stokes Equations: Saddle Point Problem Formulation

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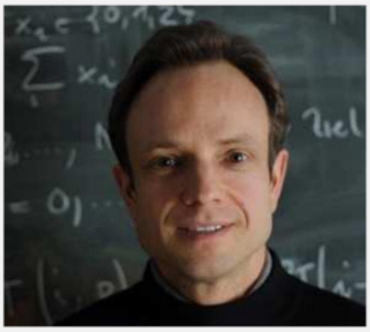


Prerequisites.

- Partial differentiation
- Lagrange multipliers

Dependencies. [Lecture → Section 1.3], [Lecture → Section 1.4], [Lecture → Section 12.2.1]

Duration: 60 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XII. Finite Elements for the Stokes Equations

12.2. The Stokes Equations

12.2.2. Saddle Point Problem Formulation

$$\mathbf{v} \in \boldsymbol{H}_0^1(\mathrm{div}\,0, \Omega): \quad \mathsf{a}(\mathbf{v}, \mathbf{w}) = \ell(\mathbf{w}) \quad \forall \mathbf{w} \in \boldsymbol{H}_0^1(\mathrm{div}\,0, \Omega)\,,$$

$\hookrightarrow$ s.p.d. ("elliptic")

$\Longleftrightarrow \quad \frac{1}{2} a(\underline{v}, \underline{v}) - \ell(\underline{v}) \longrightarrow \min$

Idea: $\quad \mathrm{div}\, \underline{v} = 0$: treat as constraint

Idea: Weak enforcement of the divergence constraint $\mathrm{div}\,\mathbf{v} = 0$ through a Lagrange multiplier

Abstract: $\to$ multiplier space

- $U, Q \mathrel{\hat=}$ real Hilbert spaces with inner products $(\cdot,\cdot)_U$, $(\cdot,\cdot)_Q$,
- $J : U \mapsto \mathbb{R}$ convex and differentiable functional,
- $\mathsf{B} : U \mapsto Q$ a continuous linear operator, defining the **constraint** through $u \in U$: $\mathsf{B}u = 0$.

linearly **constrained minimization problem**,

$$v^* \in U: \quad v^* = \underset{v \in U, \mathsf{B}v = 0}{\operatorname{argmin}}\; J(v)\,. \tag{12.2.2.2}$$

Lagrangian:

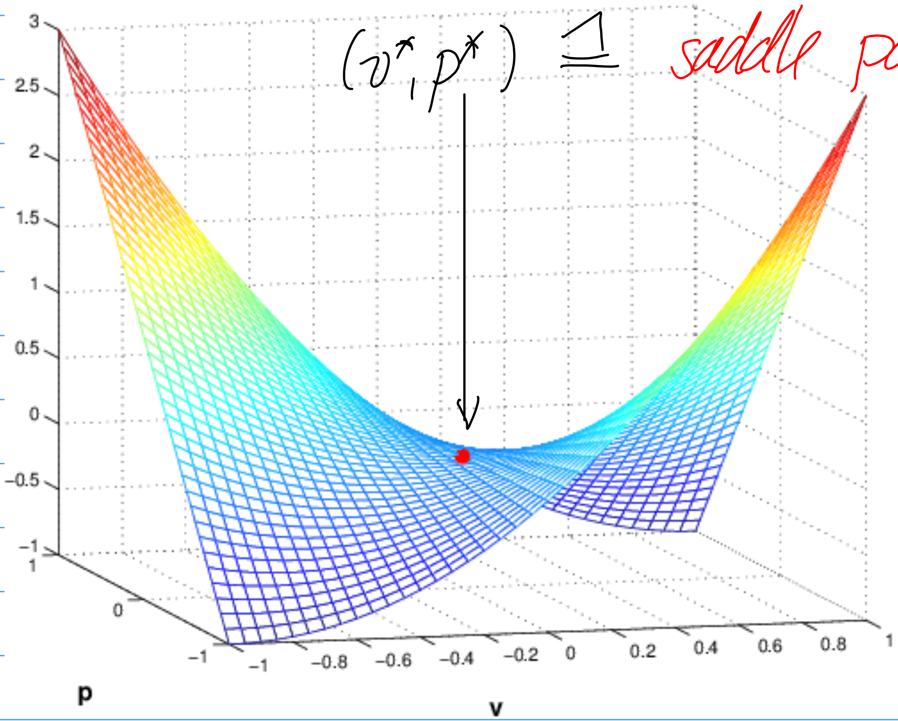
$$L(v,p) := J(v) + (p, \mathsf{B}v)_Q \,, \quad v \in U,\ p \in Q \quad \blacktriangleright \quad v^* = \underset{v\in U}{\operatorname{argmin}}\, \sup_{p\in Q} L(v,p) \,, \tag{12.2.2.3}$$

Lagrange multiplier

unconstrained min-max prb.

Lemma 12.2.2.4. Necessary conditions for existence of solution of saddle point problems →

Any solution v^* of (12.2.2.3) *will be the first component of a zero (v^*,p^*) of the derivative ("gradient") of the Lagrangian functional L:* $\underline{\mathsf{D}L(v^*,p^*) = 0}$.



$(v^*,p^*) \overset{\Delta}{=}$ saddle point : non-extremal stationary point

$$v^* \ \text{solves (12.2.2.3)} \ \Rightarrow \ \exists p^* \in U:\ \underbrace{\mathsf{D}L(v^*,p^*)}(w,q) = 0 \quad \forall w \in U,\ q \in Q \,. \tag{12.2.2.6}$$

More explicitly via:

$$\frac{\partial L}{\partial v}(v,p)(w) = \mathsf{D}J(v)(w) + (p, \mathsf{B}w)_Q \,, \qquad v,w \in U \,, \quad p \in Q \,, \tag{12.2.2.8a}$$

$$\frac{\partial L}{\partial p}(v,p)(q) = (q, \mathsf{B}v)_Q \,, \qquad p,q \in Q \,, \quad v \in U \,. \tag{12.2.2.8b}$$

→ Lagrange multiplier

$$(12.2.2.6) \ \Leftrightarrow \ \begin{aligned} \langle DJ(v^*), w\rangle \ + \ (p^*, \mathsf{B}w)_Q \ &= \ 0 \quad \forall w \in U \,, \\ (q, \mathsf{B}v^*)_Q \ &= \ 0 \quad \forall q \in Q \,. \end{aligned} \tag{12.2.2.9}$$

$\overset{\Delta}{=}$ variational saddle point problem (SPP)

§ 1.2.2.10 (Linear constrained QMP)

Now: $J(v) := \frac{1}{2}a(v,v) - \ell(v)$, $v \in \mathcal{U}$

$\uparrow$ s.p.d. blf.

$\triangleright$ $DJ(v)(w) = a(v,w) - \ell(w)$, $w \in \mathcal{U}$

Seek $v^* \in U$, $p^* \in Q$

$$\begin{aligned} \mathsf{a}(v^*, w) &+ (p^*, \mathsf{B}w)_Q &= \ell(w) \quad &\forall w \in U\,,\\ (\mathsf{B}v^*, q)_Q & &= 0 \quad &\forall q \in Q\,. \end{aligned} \tag{12.2.2.11}$$

Specialized to fluid model:

$$\mathbf{v} \in \boldsymbol{H}_0^1(\operatorname{div} 0, \Omega):\quad \int_\Omega \mathsf{D}\mathbf{v} : \mathsf{D}\mathbf{w}\, d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{w}\, d\mathbf{x} \quad \forall \mathbf{w} \in \boldsymbol{H}_0^1(\operatorname{div} 0, \Omega)\,, \tag{12.2.1.15}$$

$$\boldsymbol{H}_0^1(\operatorname{div} 0, \Omega) := \left\{\mathbf{v} \in (H_0^1(\Omega))^d : \operatorname{div} \mathbf{v} = 0\right\}. \tag{12.2.2.13}$$

$$\mathsf{B} \triangleq \operatorname{div} : \underbrace{(H_0^1(\Omega))^d}_{=:\,\mathcal{U}} \longrightarrow L^2(\Omega) =: Q$$

- the Hilbert spaces $U = (H_0^1(\Omega))^d$, $Q = L^2(\Omega)$,
- the constraint $\operatorname{div} \mathbf{v} = 0$ $\succ$ $\mathsf{B} := \operatorname{div} : U \mapsto Q$, obviously continuous,
- $J \leftrightarrow \mathbf{v} \mapsto \frac{1}{2}\int_\Omega \mu \|\mathsf{D}\mathbf{v}\|^2\, d\mathbf{x} - \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}$, a strictly convex quadratic functional $\mathbf{v} \mapsto \frac{1}{2}\mathsf{a}(\mathbf{v},\mathbf{v}) - \ell(\mathbf{v})$ ($\rightarrow$ Def. 1.2.3.2) with building blocks

$$\mathsf{a}(\mathbf{v},\mathbf{w}) := \int_\Omega \mu \|\mathsf{D}\mathbf{v}\|^2\, d\mathbf{x}\,,\quad \ell(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}\,,\quad \mathbf{v},\mathbf{w} \in U\,. \tag{12.2.2.14}$$

Seek the velocity field $\mathbf{v} \in (H_0^1(\Omega))^d$ and a Lagrange multiplier $p \in L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \mu \mathsf{D}\mathbf{v} : \mathsf{D}\mathbf{w}\, d\mathbf{x} &+ \underbrace{\int_\Omega \operatorname{div}\mathbf{w}\, p\, d\mathbf{x}}_{} &= \int_\Omega \mathbf{f}\cdot\mathbf{w}\, d\mathbf{x} \quad &\forall \mathbf{w} \in (H_0^1(\Omega))^d\,,\\ \int_\Omega \operatorname{div}\mathbf{v}\, q\, d\mathbf{x} & &= 0 \quad &\forall q \in L^2(\Omega)\,. \end{aligned} \tag{12.2.2.16}$$

the pressure

5 $\underline{w} \in (H_0^1(\Omega))^d \overset{\text{Gauss thm.}}{\Longrightarrow} \int_\Omega \operatorname{div} \underline{w} \, dx = 0$

$p + c \;\; \forall c \in \mathbb{R}$ is a pressure solution of (2.16)

Remedy as Rem. 1.8.0.14

Choose $\quad p \in L^2_*(\Omega) := \{q \in L^2(\Omega) \colon \int_\Omega q \, \mathrm{d}\boldsymbol{x} = 0\}$. (12.2.2.18)

$\longleftrightarrow$ Linear constraint on "pressure" trial/test space $L^2(\Omega)$

§ 12.2.2.20 Theory of SPPs

$$\begin{array}{ll} v \in U \\ p \in Q \end{array} : \quad \begin{array}{rcll} \mathsf{a}(v,w) + \mathsf{b}(w,p) &=& \ell(w) & \forall w \in U , \\ \mathsf{b}(v,q) &=& g(q) & \forall q \in Q . \end{array} \tag{12.2.2.21}$$

where
- U and Q are Hilbert spaces, with norms $\|\cdot\|_U$ and $\|\cdot\|_Q$, respectively,
- $\mathsf{a} : U \times U \to \mathbb{R}$ and $\mathsf{b} : U \times Q \to \mathbb{R}$ are continuous/bounded bilinear forms,
- $\ell : U \to \mathbb{R}$ and $g : Q \to \mathbb{R}$ are continuous/bounded linear forms.

$\mathcal{N}(b) := \{w \in U : b(w,q) = 0 \;\; \forall q \in Q\}$

Theorem 12.2.2.23. Ladyzhenskaya–Babuška–Brezzi conditions (LBB-conditions)

If the following two conditions are satisfied,

(i) the bilinear form a *is* $\mathcal{N}(\mathsf{b})$*-elliptic (**ellipticity on the kernel**)*

$$\exists \alpha > 0\colon \quad |\mathsf{a}(v,v)| \geq \alpha \|v\|_U^2 \quad \forall v \in \mathcal{N}(\mathsf{b}) , \tag{LBB1}$$

(ii) the bilinear form b *satisfies the **inf-sup condtion***

$$\exists \beta > 0\colon \quad \sup_{v \in U} \frac{|\mathsf{b}(v,q)|}{\|v\|_U} \geq \beta \|q\|_Q \quad \forall q \in Q , \tag{LBB2}$$

then the variational saddle point problem (12.2.2.50) *possesses a unique solution* $(v,p) \in U \times Q$*, which satisfies*

$$\|v\|_U + \|p\|_Q \leq C \left(\sup_{w \in U} \frac{|\ell(w)|}{\|w\|_U} + \sup_{w \in U} \frac{|g(q)|}{\|q\|_Q} \right) , \tag{12.2.2.24}$$

with $C > 0$ *independent of* ℓ *and* g.

Setting : $\mathcal{M} = \mathbb{R}^m$, $\mathcal{Q} = \mathbb{R}^n$ (Euclidean space)

$$\begin{aligned} \vec{v} \in \mathbb{R}^m \\ \vec{\pi} \in \mathbb{R}^n \end{aligned} : \quad \begin{aligned} \mathsf{a}(\vec{v},\vec{\eta}) + \mathsf{b}(\vec{\eta},\vec{\pi}) &= \ell(\vec{\eta}) \quad \forall \vec{\eta} \in \mathbb{R}^m , \\ \mathsf{b}(\vec{v},\vec{\omega}) &= g(\vec{\omega}) \quad \forall \vec{\omega} \in \mathbb{R}^n . \end{aligned} \tag{12.2.2.30}$$

Matrix/vector repr.

$$\begin{aligned} \exists \mathbf{A} \in \mathbb{R}^{m,m}: \quad & \mathsf{a}(\vec{v},\vec{\eta}) = \vec{\eta}^\top \mathbf{A} \vec{v} \quad \forall \vec{\eta}, \vec{v} \in \mathbb{R}^m , \\ \exists \mathbf{B} \in \mathbb{R}^{n,m}: \quad & \mathsf{b}(\vec{v},\vec{\omega}) = \vec{\omega}^\top \mathbf{B} \vec{v} \quad \forall \vec{v} \in \mathbb{R}^m,\ \vec{\omega} \in \mathbb{R}^n , \end{aligned}$$

$$\exists \vec{\varphi} \in \mathbb{R}^m: \quad \ell(\vec{\eta}) = \vec{\eta}^\top \vec{\varphi} \quad \forall \vec{\eta} \in \mathbb{R}^m \quad , \quad \exists \vec{\gamma} \in \mathbb{R}^m: \quad g(\vec{\omega}) = \vec{\omega}^\top \vec{\gamma} \quad \forall \vec{\omega} \in \mathbb{R}^n ,$$

LSE :

(2.30) $\Rightarrow$

$$\begin{aligned} \vec{v} \in \mathbb{R}^m \\ \vec{\pi} \in \mathbb{R}^n \end{aligned} : \quad \begin{aligned} \mathbf{A}\vec{v} + \mathbf{B}^\top \vec{\pi} &= \vec{\varphi} \quad , \\ \mathbf{B}\vec{v} &= \vec{\gamma} \quad , \end{aligned} \tag{12.2.2.32}$$

$$\begin{array}{c} m \\ n \end{array} \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B}^\top \\ \hline \mathbf{B} & \mathbf{O} \end{array}\right] \begin{bmatrix} \vec{v} \\ \vec{\pi} \end{bmatrix} = \begin{bmatrix} \vec{\varphi} \\ \vec{\gamma} \end{bmatrix} . \tag{12.2.2.33}$$

$\mathcal{N}(\mathsf{b}) \longleftrightarrow \mathcal{N}(\mathbf{B}) \subset \mathbb{R}^m$ [nullspace of B]

LBB2 $\Rightarrow$
$$\exists \beta > 0: \quad \sup_{\vec{\eta} \in \mathbb{R}^m} \frac{\vec{\eta}^\top \mathbf{B}^\top \vec{\omega}}{\|\vec{\eta}\|_2} \geq \beta \|\vec{\omega}\|_2 \quad \forall \vec{\omega} \in \mathbb{R}^n . \tag{12.2.2.35}$$

$\Rightarrow$ $\mathcal{N}(B^T) = \{0\}$: necessary for uniqueness of $\vec{\pi}$

$\Rightarrow$ B full rank n : necessarily $n \leq m$

$\Rightarrow$ $\mathcal{R}(B) = \mathbb{R}^n$

$\Rightarrow \exists \vec{v}_g \in \mathbb{R}^m: \ B\vec{v}_g = \vec{\gamma}$

Set: $\vec{v} = \vec{v}_g + \vec{\mu}, \ \vec{\mu} \in N(B)$

LBBI: $\vec{\eta}^T A \vec{\eta} \geq \alpha \|\vec{\eta}\|_2^2 \quad \forall \vec{\eta} \in N(B)$

$\Rightarrow A_{|N(B)}$ regular

$\cdot \vec{\eta}^T, B\vec{\eta}=0: \ A\vec{\mu} + B^T\vec{\pi} = \vec{\varphi} - A\vec{v}_g$

$\vec{\eta}^T A \vec{\mu} = \vec{\eta}^T(\vec{\varphi} - A\vec{v}_g) \quad \forall \vec{\eta} \in N(B)$

unique solution $\vec{\mu} \in N(B)$

(*) $B^T\vec{\pi} = \vec{\varphi} - A(\vec{v}_g + \vec{\mu}) \in N(B)^\perp$

LA: $N(B)^\perp = \mathcal{R}(B^T)$

(*) has consistent r.h.s. $\qquad \exists \vec{\pi}\,!$

Apply to Stokes SPP:

$$\begin{aligned}
\int_\Omega \mu\, \mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{w}\, \mathrm{d}\mathbf{x} + \int_\Omega \operatorname{div} \mathbf{w}\, p\, \mathrm{d}\mathbf{x} &= \int_\Omega \mathbf{f} \cdot \mathbf{w}\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{w} \in (H_0^1(\Omega))^d, \\
\int_\Omega \operatorname{div} \mathbf{v}\, q\, \mathrm{d}\mathbf{x} &= 0 \quad \forall q \in L_*^2(\Omega),
\end{aligned} \tag{12.2.2.19}$$

$$\begin{aligned}
v \in U &: \quad \mathsf{a}(v,w) + \mathsf{b}(w,p) = \ell(w) \quad \forall w \in U, \\
p \in Q &: \quad \mathsf{b}(v,q) = g(q) \quad \forall q \in Q.
\end{aligned} \tag{12.2.2.21}$$

where
- U and Q are Hilbert spaces, with norms $\|\cdot\|_U$ and $\|\cdot\|_Q$, respectively,
- $\mathsf{a} : U \times U \to \mathbb{R}$ and $\mathsf{b} : U \times Q \to \mathbb{R}$ are continuous/bounded bilinear forms,
- $\ell : U \to \mathbb{R}$ and $g : Q \to \mathbb{R}$ are continuous/bounded linear forms.

- Hilbert spaces $U := (H_0^1(\Omega))^d$, $Q := L_*^2(\Omega)$,
- bilinear form $\mathsf{a} \leftrightarrow (\mathbf{v},\mathbf{w}) \mapsto \int_\Omega \mu\, \mathsf{D}\mathbf{v}(\boldsymbol{x}) : \mathsf{D}\mathbf{w}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$,
- bilinear form $\mathsf{b} \leftrightarrow (\mathbf{v},q) \mapsto \int_\Omega \operatorname{div}\mathbf{v}(\boldsymbol{x})\, q(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$,
- and r.h.s. linear forms $\ell \leftrightarrow \mathbf{w} \mapsto \int_\Omega \mathbf{f}(\boldsymbol{x}) \cdot \mathbf{w}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$, and $g \equiv 0$.

• LBB 1 on $\mathcal{U}$: by P.F. inequ. &

$$\sum_{i=1}^d \int_\Omega \mathbf{grad}\, v_i \cdot \mathbf{grad}\, w_i\, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathsf{D}\mathbf{v} : \mathsf{D}\mathbf{w}\, \mathrm{d}\boldsymbol{x}\,, \quad \mathbf{v},\mathbf{w} \in \left(H_0^1(\Omega)\right)^d, \tag{12.2.2.37}$$

• LBB 2 : $\exists \beta > 0$: $\sup_{v \in U} \frac{|\mathsf{b}(v,q)|}{\|v\|_U} \geq \beta \|q\|_Q \quad \forall q \in Q$ (LBB2)

$$\sup_{w \in (H_0^1(\Omega))^d} \frac{\int_\Omega \operatorname{div} w \cdot q\, dx}{\|w\|_{H^1}} \geq \beta \|q\|_{L^2} \quad \forall q \in L_*^2(\Omega)$$

Theorem 12.2.2.38. Existence of stable velocity potentials

$$\exists \mathsf{C} = \mathsf{C}(\Omega) > 0: \quad \forall q \in L_*^2(\Omega): \quad \exists \mathbf{v} \in (H_0^1(\Omega))^d: \quad q = \operatorname{div}\mathbf{v} \quad \wedge \quad \|\mathbf{v}\|_{H^1(\Omega)} \leq \mathsf{C}\|q\|_{L^2(\Omega)}\,.$$

$$\sup_{\mathbf{w} \in (H_0^1(\Omega))^d} \frac{\left|\int_\Omega \operatorname{div}\mathbf{w}(\boldsymbol{x})\, q(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right|}{\|\mathbf{w}\|_{H^1(\Omega)}} \geq \frac{\left|\int_\Omega \operatorname{div}(\mathbf{v}(q))(\boldsymbol{x})\, q(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right|}{\|\mathbf{v}(q)\|_{H^1(\Omega)}} \geq \frac{\int_\Omega q^2(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}}{\mathsf{C}\|q\|_{L^2(\Omega)}} = \frac{1}{\mathsf{C}}\|q\|_{L^2(\Omega)}\,,$$

Theorem 12.2.2.40. Existence and uniqueness of weak solutions of Stokes problem

The linear variational saddle point problem (12.2.2.19) *("Stokes problem") has a unique solution* $(\mathbf{v},p) \in \boldsymbol{H}_0^1(\operatorname{div} 0,\Omega) \times L_*^2(\Omega)$, *which satisfies*

$$\exists \mathsf{C} = \mathsf{C}(\Omega) > 0: \quad \|\mathbf{v}\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq \mathsf{C}\|\mathbf{f}\|_{L^2(\Omega)}\,. \tag{12.2.2.41}$$

Stability estimate