ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 12.2.3: The Stokes Equations: Stokes System of PDEs

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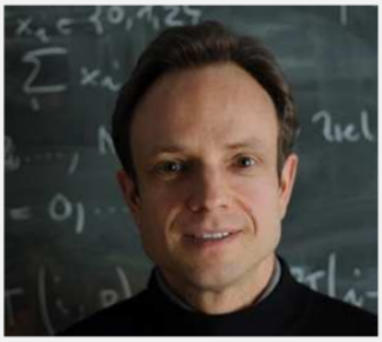

Prerequisites.

- Multi-dimensional calculus and vector analysis

Dependencies. [Lecture → Section 1.5.3], [Lecture → Section 1.9], [Lecture → Section 12.2.2]

Duration: 15 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XII. Finite Elements for the Stokes Equations

12.2. The Stokes Equation

12.2.3 Stokes System of PDEs

velocity   pressure   essential b.c.: no-slipe

$$\begin{aligned}
\int_\Omega \mu \mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{w}\,\mathrm{d}\mathbf{x} + \int_\Omega \operatorname{div}\mathbf{w}\, p\,\mathrm{d}\mathbf{x} &= \int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} && \forall \mathbf{w} \in (H_0^1(\Omega))^d\,, \\
\int_\Omega \operatorname{div}\mathbf{v}\, q\,\mathrm{d}\mathbf{x} &= 0 && \forall q \in L_*^2(\Omega)\,.
\end{aligned} \qquad (12.2.2.19)$$

Goal: Extract PDE by reverse i.b.p.
(Green's formula)

div∘grad = Δ

$$\int_\Omega \mu \nabla\mathbf{v} : \nabla\mathbf{w}\,\mathrm{d}\mathbf{x} = \mu \sum_{i=1}^{d} \int_\Omega \mathbf{grad}\, v_i \cdot \mathbf{grad}\, w_i\,\mathrm{d}\mathbf{x} = -\mu \sum_{i=1}^{d} \int_\Omega \Delta v_i\, w_i\,\mathrm{d}\mathbf{x}\,,$$

$$\int_\Omega \operatorname{div}\mathbf{w}\, p\,\mathrm{d}\mathbf{x} = -\int_\Omega \mathbf{grad}\, p \cdot \mathbf{w}\,\mathrm{d}\mathbf{x}\,,$$

Assumption 12.2.3.1. Extra smoothness of velocity and pressure

For the solution $(\mathbf{v}, p)$ of (12.2.2.19) we assume

$$\mathbf{v} = \left(H^2(\Omega)\right)^d \quad \text{and} \quad u \in H^1(\Omega)\,.$$

$$-\mu \sum_{i=1}^{d} \int_\Omega \Delta v_i \, w_i \, dx - \int_\Omega \mathbf{grad}\, p \cdot \mathbf{w} \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{w} \, dx \quad \forall \mathbf{w} \in (H_0^1(\Omega))^d ,$$
$$\int_\Omega \operatorname{div} \mathbf{v} \, q \, dx = 0 \quad \forall q \in L_*^2(\Omega) . \tag{12.2.3.3}$$

$$\int_\Omega \left(-\mu \underbrace{\begin{bmatrix} \Delta v_1 \\ \vdots \\ \Delta v_d \end{bmatrix}}_{\Delta \mathbf{v} \,\hat{=}\, \text{vector Laplacian}} - \mathbf{grad}\, p - \mathbf{f} \right) \cdot \mathbf{w} \, dx = 0 \quad \forall \mathbf{w} \in \left(H_0^1(\Omega) \right)^d . \tag{12.2.3.4}$$

(12.2.2.19) $\Rightarrow$

$$\boxed{\begin{aligned} -\mu \boldsymbol{\Delta} \mathbf{v} - \mathbf{grad}\, p &= \mathbf{f} \\ \operatorname{div} \mathbf{v} &= 0 \quad \text{in} \quad \Omega , \\ \int_\Omega p \, dx &= 0 \\ \mathbf{v} &= 0 \quad \text{on} \quad \partial\Omega . \end{aligned}} \tag{12.2.3.5}$$

← no-slip b.c.

Rem.: 12.2.3.5 (Pressure Poisson Equ.)

$\operatorname{div} \circ \Delta = \Delta \circ \operatorname{div}$:

div · (12.2.3.5) ▶ $-\mu \operatorname{div} \boldsymbol{\Delta} \mathbf{v} + \operatorname{div} \mathbf{grad}\, p = \operatorname{div} \mathbf{f}$ in Ω ,

▶ $-\mu \Delta(\operatorname{div} \mathbf{v}) + \Delta p = \operatorname{div} \mathbf{f}$ in Ω ,

$\xrightarrow{\operatorname{div} \mathbf{v} = 0}$ ▶ $\Delta p = \operatorname{div} \mathbf{f}$.

1) Solve : $\Delta p = \operatorname{div} \mathbf{f}$ in Ω

What's the problem? Missing b.c.

2) Solve: $\boldsymbol{\Delta} \mathbf{v} = \mathbf{f} - \mathbf{grad}\, p$ in Ω
$\mathbf{v} = 0$ on $\partial\Omega$