ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 12.3.1: Pressure Instability

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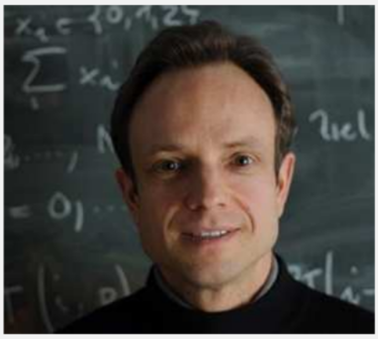


Dependencies. [Lecture → § 12.3.0.1], [Lecture → Section 2.6], [Lecture → Section 12.2.2]

Duration: 40 minutes

⚠ Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XII. Finite Elements for the Stokes Equations

12.3. Galerkin Discretization of the Stokes Saddle Point Problem

$\underline{v} \in (H_0^1(\Omega))^d$, $p \in L_*^2(\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \mu \mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{w}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{div}\mathbf{w}\,p\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}\boldsymbol{x} \quad \forall \mathbf{w} \in (H_0^1(\Omega))^d\,, \\ \int_\Omega \operatorname{div}\mathbf{v}\,q\,\mathrm{d}\boldsymbol{x} &= 0 \quad \forall q \in L_*^2(\Omega)\,. \end{aligned} \tag{12.2.2.19}$$

$$\Updownarrow$$

$$\begin{aligned} \mathbf{v} \in U &:= \left(H_0^1(\Omega)\right)^d \\ p \in Q &:= L_*^2(\Omega) \end{aligned} : \quad \begin{aligned} \mathsf{a}(\mathbf{v},\mathbf{w}) + \mathsf{b}(\mathbf{w},p) &= \ell(\mathbf{w}) \quad \forall \mathbf{w} \in U\,, \\ \mathsf{b}(\mathbf{v},q) &= 0 \quad \forall q \in Q\,. \end{aligned} \tag{12.3.0.2}$$

§ 12.3.0.1 Galerkin discretization of saddle point problems

Step ①
$$\begin{aligned} U &\longrightarrow U_h \subset U \\ Q &\longrightarrow Q_h \subset Q \end{aligned} \quad \text{in (2.19)}$$

$$N := \dim U_h\,,\ M := \dim Q_h < \infty$$

DVP:

$$\begin{aligned} \mathbf{v}_h \in U_h &: \quad \mathsf{a}(\mathbf{v}_h, \mathbf{w}_h) + \mathsf{b}(\mathbf{w}_h, p_h) = \ell(\mathbf{w}_h) \quad \forall \mathbf{w}_h \in U_h , \\ p_h \in Q_h &: \quad \mathsf{b}(\mathbf{v}_h, q_h) \qquad\quad\ \ 0 \qquad\ \ = 0 \qquad\quad \forall q_h \in Q_h . \end{aligned} \tag{12.3.0.4}$$

Step ② : Choose bases $\{b_h^i\}_{i=1}^N$ of U_h
$\{\beta^j\}_{j=1}^M$ of Q_h

+ expansion : $v_h = \sum_{i=1}^N v_i b_h^i$

$p_h = \sum_{j=1}^M \pi_j \beta_h^j$

▷ LSE

saddle point matrix
[symmetric, not s.p.d.]

saddle point LSE:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{\pi} \end{bmatrix} = \begin{bmatrix} \vec{\varphi} \\ \mathbf{0} \end{bmatrix}, \tag{12.3.0.7}$$

$$\mathbf{A} := \left[\mathsf{a}(\mathbf{b}_h^j, \mathbf{b}_h^i)\right]_{i,j=1}^N = \left[\int_\Omega \mu\, \mathsf{D}\mathbf{b}_h^j(\boldsymbol{x}) : \mathsf{D}\mathbf{b}_h^i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right]_{i,j=1}^N \in \mathbb{R}^{N,N}, \tag{12.3.0.8a}$$

$$\mathbf{B} := \left[\mathsf{b}(\mathbf{b}_h^j, \beta_h^i)\right]_{\substack{1\le i\le M\\1\le j\le N}} = \left[\int_\Omega \operatorname{div} \mathbf{b}_h^j(\boldsymbol{x})\, \beta_h^i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right]_{\substack{1\le i\le M\\1\le j\le N}} \in \mathbb{R}^{M,N}, \tag{12.3.0.8b}$$

$$\vec{\varphi} := \left[\ell(\mathbf{b}_h^j)\right]_{j=1}^N = \left[\int_\Omega \mathbf{f}(\boldsymbol{x}) \cdot \mathbf{b}_h^j(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right]_{j=1}^N \in \mathbb{R}^N. \tag{12.3.0.8c}$$

12.3.1. Pressure Instability

Simple setting of Sect. 3.1

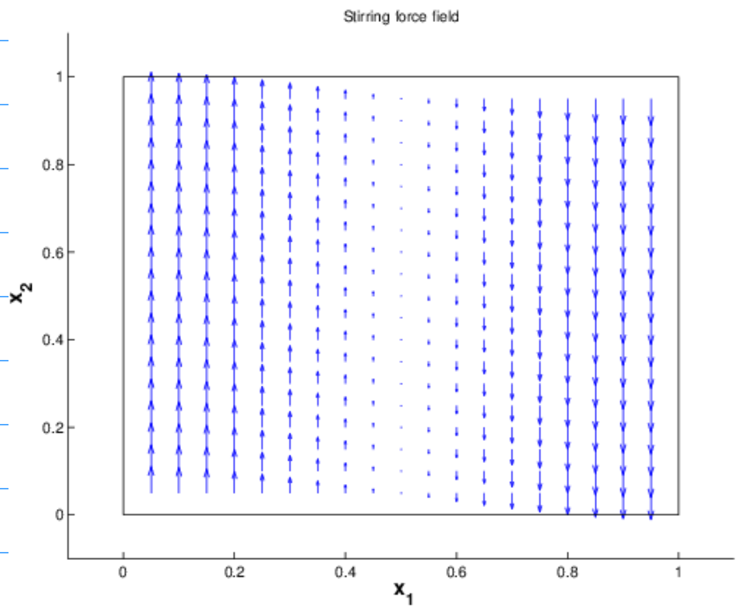
V.P.: $u \in V_0$: $a(u,v) = \ell(v) \quad \forall v \in V_0$

$\hookrightarrow$ s.p.d. $\Rightarrow$ induces $\|\cdot\|_a$

Gal. disc.: $V_0 \longrightarrow V_{0,h} \subset V_0$

Cea's lemma: $\|u - u_h\|_a \leq \inf_{v_h \in V_{0,h}} \|u - v_h\|_a$

Exp. 12.3.1.1: Q1-Q1 FEM on tensor-product mesh

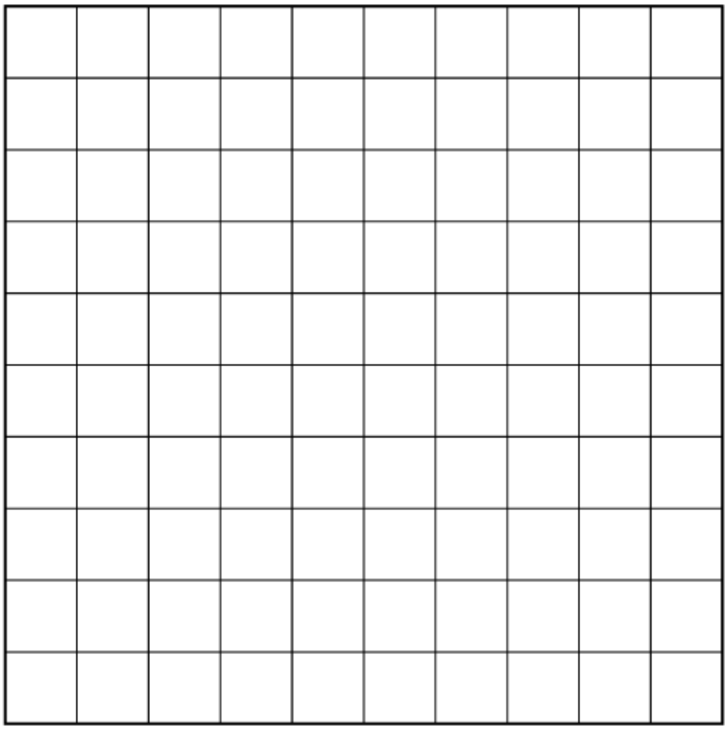


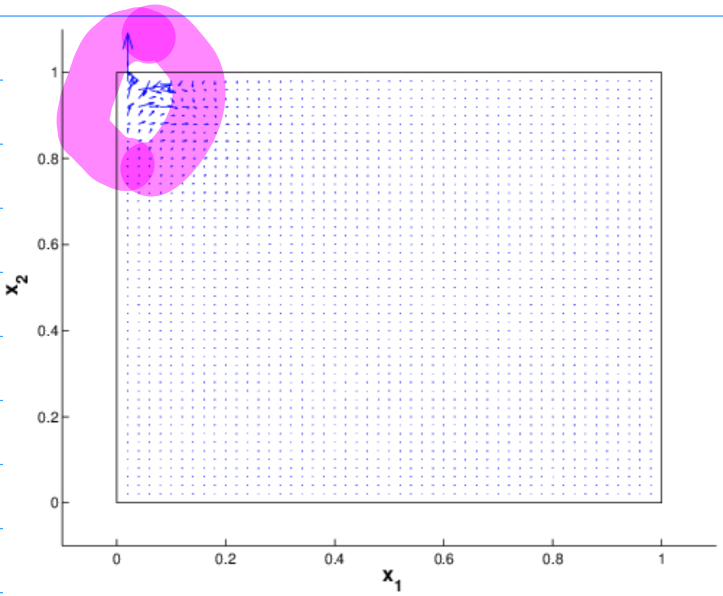
- on the unit square $\Omega =]0,1[^2$,
- with $\mu \equiv 1$, and
- with driving force field $\mathbf{f} = \cos(\pi x_1)\begin{bmatrix}0\\1\end{bmatrix}$,

◁ Right-hand side force field $\mathbf{f}$

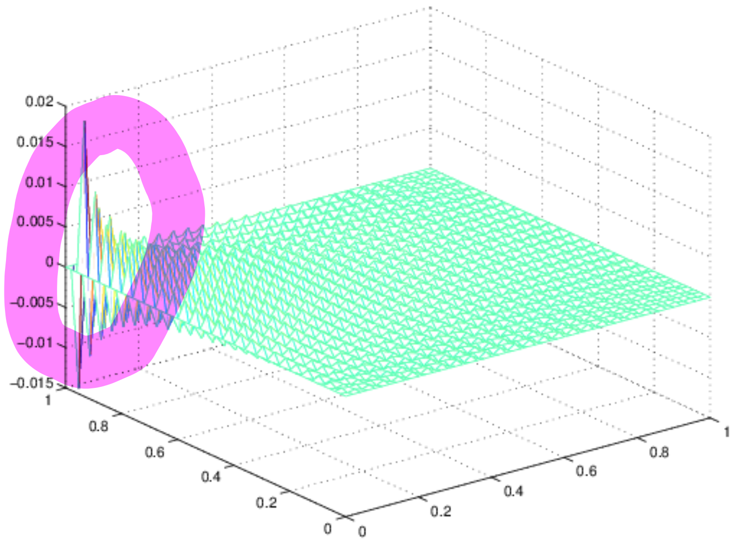
- $n \times n$ tensor-product quadrilateral mesh $\mathcal{M}$ ▷
- with $U_h := (\mathcal{S}^0_{1,0}(\mathcal{M}))^2 \subset (H^1_0(\Omega))^2$,
- and $Q_h := \mathcal{S}^0_1(\mathcal{M}) \subset L^2(\Omega)$,

p.w. bilinear





velocity solution $\mathbf{v}_h$



pressure solution p_h

→ spurious oscillations

Back $(M+N) \times (M+N)$

saddle point LSE: $$N \updownarrow \begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{\pi} \end{bmatrix} = \begin{bmatrix} \vec{\varphi} \\ \mathbf{0} \end{bmatrix}, \quad \underset{M}{\longleftrightarrow} \tag{12.3.0.7}$$

$\Rightarrow$ If $\mathcal{N}(B^T) \neq \{0\}$ $\Rightarrow$ $\vec{\pi}$ not unique

$M > N \Rightarrow \mathcal{N}(B^T) = \{0\}$

$\dim U_h \geq \dim Q_h$ is a *necessary* condition for the uniqueness of the pressure solution p_h of the discrete saddle point variational problem (12.3.0.4).

FEM with discontinuous pressure:

$$\mathcal{S}_p^{-1}(\mathcal{M}) := \{q_h \in L^2(\Omega) : q_h|_K \in \mathcal{P}_p(\mathbb{R}^d)\}, \quad \mathcal{M} \text{ a mesh of } \Omega \subset \mathbb{R}^d. \tag{12.3.1.2}$$

Need $Q_h \subset L^2_*(\Omega) \longrightarrow Q_h := \mathcal{S}_{p,*}^{-1}(\mathcal{M})$

Ex. 12.3.1.14 (P1-P0 FEM)

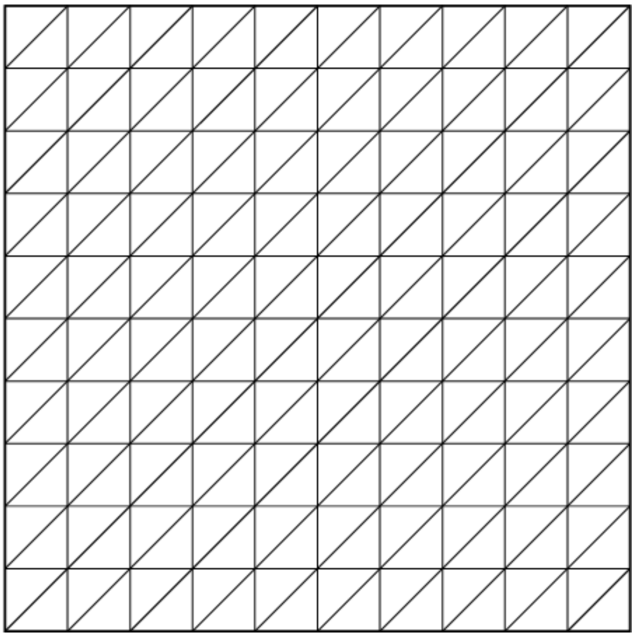
"Tensor-product" triangular mesh of $]0,1[^2$ ▷

Finite element spaces for (12.2.2.19),

$$U_h := (\mathcal{S}_{1,0}^0(\mathcal{M}))^2,$$
$$Q_h := \mathcal{S}_{0,*}^{-1}(\mathcal{M}) \quad (\mathcal{M}\text{-piecewise constants}).$$

$$\dim U_h = 2(n-1)^2 \quad , \quad \dim Q_h = 2n^2 - 1.$$

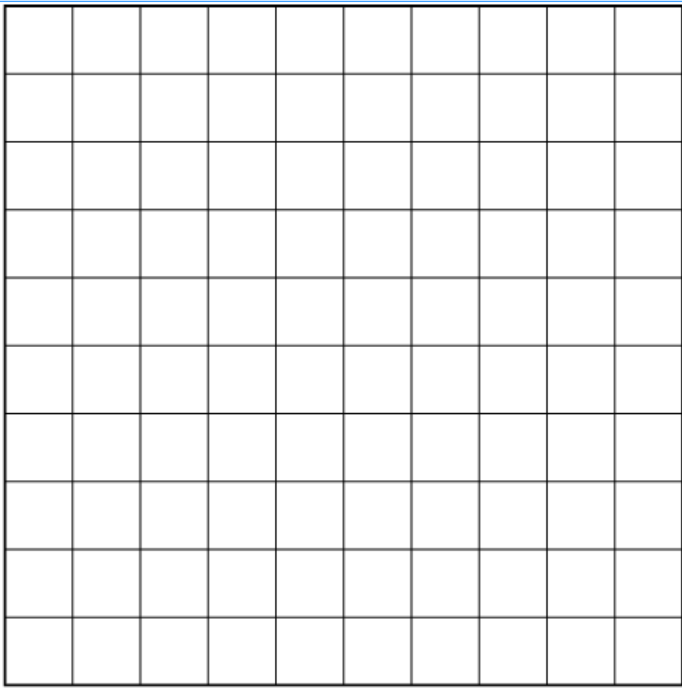
$2\#\{\text{interior nodes}\}$ $\qquad$ $\#\mathcal{M} - 1$



$\dim U_h < \dim Q_h \Rightarrow$ Singular LSE

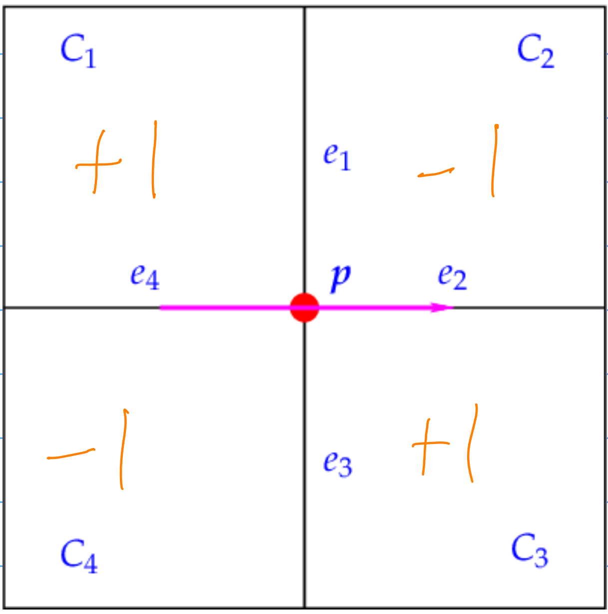
Ex. 12.3.1.5 (Q1 - P0 FEM)

- $\mathcal{M}$ is a uniform tensor product mesh of the unit square domain $\Omega :=]0,1[^2$ ▷
- velocity space $U_h = (\mathcal{S}^0_{1,0}(\mathcal{M}))^2$
- pressure space $Q_h = \mathcal{S}^{-1}_0(\mathcal{M}) \cap L^2_*(\Omega)$



$$\dim U_h = 2(n-1)^2 \quad , \quad \dim Q_h = n^2 - 1 \,.$$

▶ $\dim Q_h < \dim U_h$ for $n \geq 4$.



← p_h - cell values

$$\int_\Omega \operatorname{div} \underline{b}_h^p \cdot p_h \, dx = 0 \, !$$

Checkerboard mode ▷

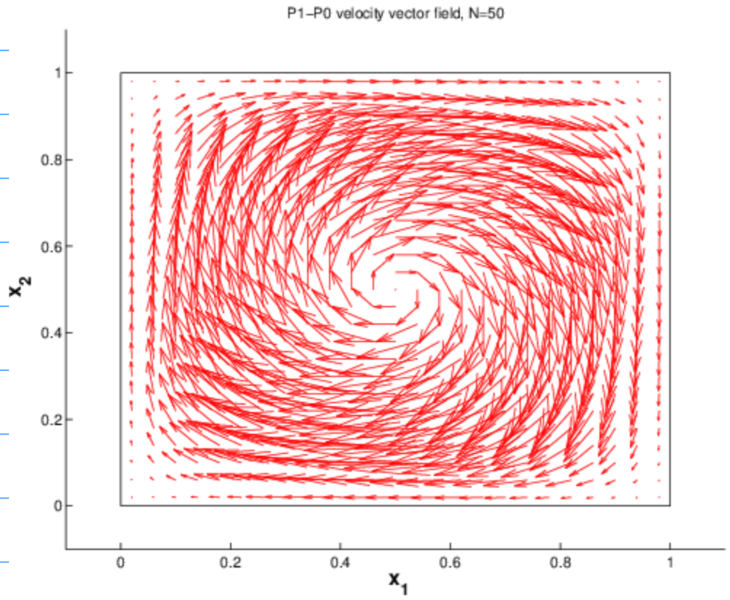
▷ $\in \mathcal{N}(B^T)$

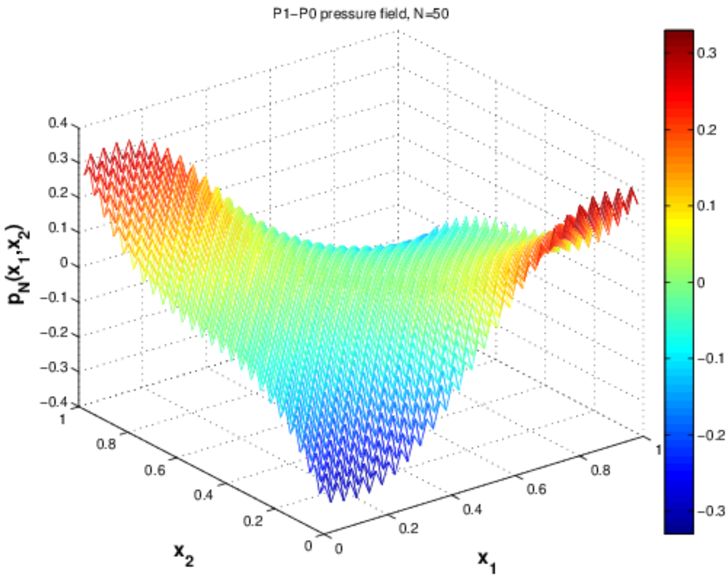
→ spurious oscillations !

$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1
-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$
$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1
-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$
$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1
-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$
$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1
-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$
$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1
-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$

Exp. 12.3.17 (P1-P0 quadrilateral FEM)

$h = 50$:





ok!

bad!