ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 12.3.2: Stable Galerkin Discretization of Stokes Saddle Point Problem

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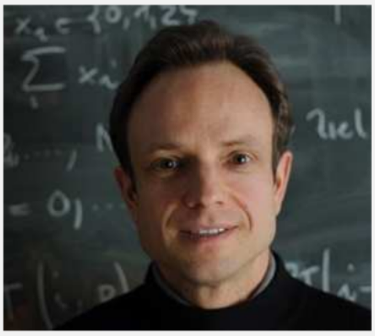


Dependencies. [Lecture → Section 2.2], [Lecture → Section 2.6], [Lecture → Section 3.1], [Lecture → Section 3.2.1], [Lecture → Section 12.2.2]

Duration: 30 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XII. Finite Elements for the Stokes Equations

12.3. Galerkin Discretization of the Stokes Saddle Point Problem

$\underline{v} \in (H_0^1(\Omega))^d$, $p \in L_*^2(\Omega)$ s.t.

$$
\begin{aligned}
\int_\Omega \mu \mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{w}\, \mathrm{d}\mathbf{x} \;+\; \int_\Omega \operatorname{div} \mathbf{w}\, p \,\mathrm{d}\mathbf{x} &= \int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} \quad \forall \mathbf{w} \in (H_0^1(\Omega))^d\,, \\
\int_\Omega \operatorname{div} \mathbf{v}\, q \,\mathrm{d}\mathbf{x} &= 0 \quad \forall q \in L_*^2(\Omega)\,.
\end{aligned}
\tag{12.2.2.19}
$$

$$\Updownarrow$$

$$
\begin{matrix} \mathbf{v} \in U := \left(H_0^1(\Omega)\right)^d \\ p \in Q := L_*^2(\Omega) \end{matrix} :
\quad
\begin{aligned}
\mathsf{a}(\mathbf{v},\mathbf{w}) + \mathsf{b}(\mathbf{w},p) &= \ell(\mathbf{w}) \quad \forall \mathbf{w} \in U\,, \\
\mathsf{b}(\mathbf{v},q) &= 0 \quad \forall q \in Q\,.
\end{aligned}
\tag{12.3.0.2}
$$

12.3.1. Pressure Instability

12.3.2. Stable Galerkin Discretization of Stokes Saddle Point Problem

$\operatorname{div} \mathbf{v}_h$ must be "large enough to fix the pressure" $p_h \in Q_h$

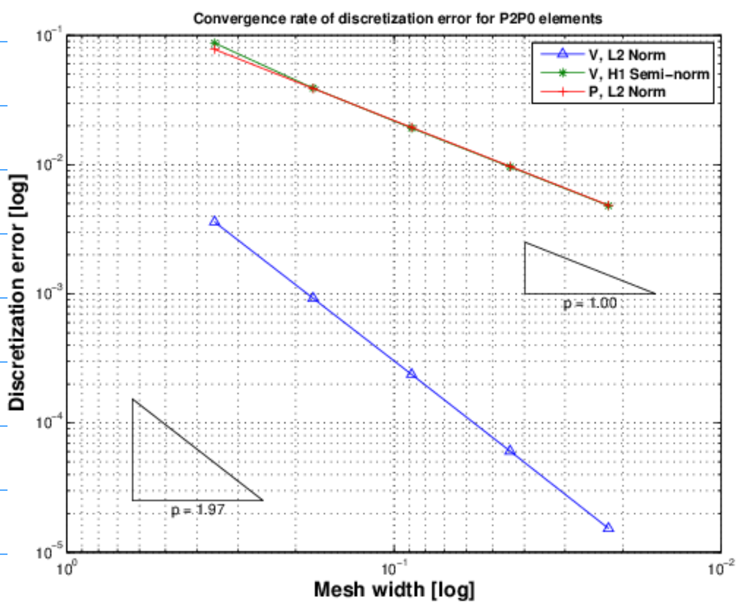
→ Enlarge $\mathcal{U}_h$, e.g., by raising p

Exp.: 12.3.21 (P2-P0 FEM)

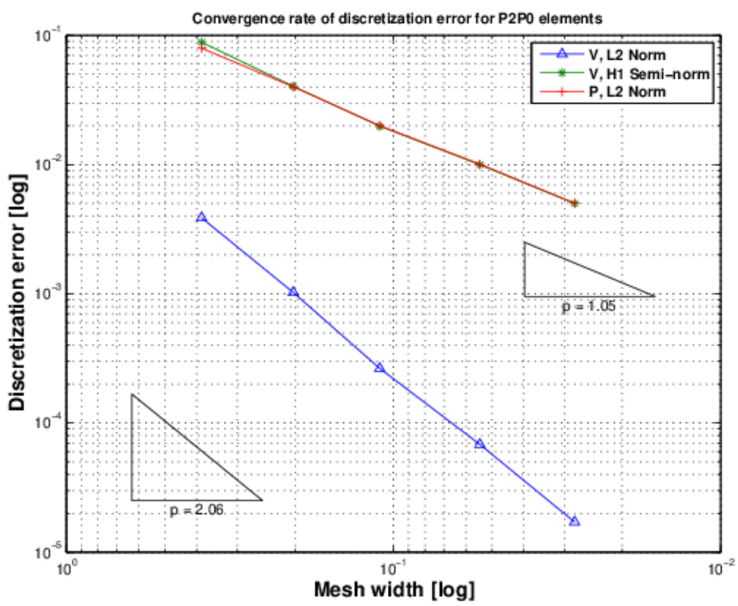
- $\Omega =]0,1[^2$, $\mathbf{v}(\boldsymbol{x}) = (\cos(\pi/2(x_1+x_2)), -\cos(\pi/2(x_1+x_2)))^T$, $p(\boldsymbol{x}) = \sin(\pi/2(x_1-x_2))$, $\mathbf{f}$ and inhomogeneous Dirichlet boundary values for $\mathbf{v}$ accordingly
- Sequence of (a) uniform triangular meshes, created by regular refinement,
  (b) randomly perturbed meshes from (a) (still uniformly shape-regular & quasi-uniform).
- "P2-P0-scheme": velocity finite element space $U_h = (\mathcal{S}^0_{2,0}(\mathcal{M}))^2$ (continuous, piecewise quadratic → Section 2.6.1, Ex. 2.6.1.2), pressure finite element space $Q_h = \mathcal{S}^{-1}_{0,*}(\mathcal{M}) := \mathcal{S}^{-1}_0(\mathcal{M}) \cap L^2_*(\Omega)$ (piecewise constant functions).

Monitored: Error norms $\|\mathbf{v}-\mathbf{v}_h\|_{H^1(\Omega)}$, $\|\mathbf{v}-\mathbf{v}_h\|_{L^2(\Omega)}$, $\|p-p_h\|_{L^2(\Omega)}$

$\downarrow$ $O(h)$ $\quad$ $\downarrow$ $O(h^2)$ $\quad$ $\downarrow$ $O(h)$ $\quad$ OK!



Uniform regular refinement



Randomly perturbed "regular" meshes

Stability: for linear v.p

$\exists C>0$: $\|$ discrete sol. $\| \leq C \|$ r.h.s = data $\|$ $\forall$ data

Stokes: $\|v_h\|_{H^1(\Omega)} + \|p_h\|_{L^2(\Omega)}$ $\cancel{\|f\|_{L^2(\Omega)}}$

weakest norm for continuous r.h.s $\Leftarrow$ dual norm $\sup_{w \in (H_0^1(\Omega))^2} \frac{|\int_\Omega f v \, dx|}{\|w\|_{H^1(\Omega)}}$

[Cauchy Schwarz $\leq \|f\|_{L^2(\Omega)}$]

LVP: $u \in V_0$: $a(u,v) = \ell(v)$ $\forall v \in V_0$

$\hookrightarrow$ s.p.d. $\Rightarrow$ induces $\|\cdot\|_a$

$$\|u_h\|_a \leq \sup_{v \in V_0} \frac{|\ell(v)|}{\|v\|_a}$$

["built-in stability"]

$$C_\ell := \sup_{\mathbf{w}\in U} \frac{|\ell(\mathbf{w})|}{\|\mathbf{w}\|_a} < \infty \quad \Leftrightarrow \quad \underbrace{|\ell(\mathbf{w})| \leq C_\ell \|\mathbf{w}\|_a \quad \forall \mathbf{w} \in U}_{\text{continuity of } \ell} . \tag{12.3.2.6}$$

indep. of $h_\mathcal{M}$!

Definition 12.3.2.7. Stable finite element pair

A pair of finite element spaces $U_h \subset \boldsymbol{H}_0^1(\Omega)$, $Q_h \subset L_*^2(\Omega)$ is a **stable finite element pair**, if the solution $(\mathbf{v}_h, p_h)$ of the discrete saddle point problem (12.3.0.4) satisfies

$$\exists C > 0\colon \quad \|\mathbf{v}_h\|_a + \|p_h\|_{L^2(\Omega)} \leq C \sup_{\mathbf{w}\in(H_0^1(\Omega))^d} \frac{|\int_\Omega \mathbf{f}(\boldsymbol{x})\mathbf{w}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}|}{\|\mathbf{w}\|_a} \quad \forall \mathbf{f} \in L^2(\Omega)\,,$$

where the constant $C > 0$ may depend only on Ω, the coefficient μ, and the shape regularity measure ($\rightarrow$ Def. 3.3.2.20) of $\mathcal{M}$.

§ 12.3.2.8 : Stability of velocity solution

$$\begin{array}{ll} \mathbf{v}_h \in U_h & : \quad \mathsf{a}(\mathbf{v}_h,\mathbf{w}_h) + \mathsf{b}(\mathbf{w}_h,p_h) = \ell(\mathbf{w}_h) \quad \forall \mathbf{w}_h \in U_h\,, \\ p_h \in Q_h & \qquad \mathsf{b}(\mathbf{v}_h,q_h) = 0 \quad \forall q_h \in Q_h\,, \end{array} \tag{12.3.0.4}$$

In 1st equ.: $\underline{w}_h := \underline{v}_h$

$\triangleright$ [2nd equ.] $a(\underline{v}_h, \underline{v}_h) = \ell(\underline{v}_h)$

(*) $\quad \|\underline{v}_h\|_a^{\cancel{2}} \leq \sup_{\underline{w} \in \mathcal{U}} \dfrac{|\ell(\underline{w})|}{\|\underline{w}\|} \cdot \cancel{\|\underline{v}_h\|_a}$

$\hat{=}$ Unconditional stability of $\underline{v}_h$!

§ 12.3.2.11 Stability of p_h: inf-sup condition

Goal:

$$\exists C > 0:\quad \|p_h\|_{L^2(\Omega)} \leq C \sup_{\mathbf{w}_h \in U_h} \frac{|\ell(\mathbf{w})|}{\|\mathbf{w}\|_{\mathsf{a}}} = C \sup_{\mathbf{w} \in (H_0^1(\Omega))^d} \frac{|\int_\Omega \mathbf{f}(\boldsymbol{x})\mathbf{w}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}|}{\|\mathbf{w}\|_{\mathsf{a}}} \quad \forall \mathbf{f} \in L^2(\Omega)\,. \tag{12.3.2.12}$$

$$b(\underline{w}_h, p_h) = \ell(\underline{w}_h) - a(\underline{v}_h, \underline{w}_h) \quad \forall \underline{w}_h \in \mathcal{U}_h$$

If $\exists C_\beta > 0$: $\quad \|p_h\|_{L^2(\Omega)} \leq C_\beta \sup_{\underline{w}_h \in \mathcal{U}_h} \dfrac{b(\underline{w}_h, p_h)}{\|\underline{w}_h\|_a}$

$$\triangleright \|p_h\| \leq C_\beta \sup_{\underline{w}_h \in \mathcal{U}_h} \frac{|\ell(\underline{w}_h) - a(\underline{v}_h, \underline{w}_h)|}{\|\underline{w}_h\|_a}$$

$$\overset{(*)}{\leq} C_\beta \cdot 2 \cdot \sup \frac{|\ell(\underline{w})|}{\|\underline{w}\|_a}$$

Theorem 12.3.2.15. inf-sup condition

The finite element spaces $U_h \subset \mathbf{H}_0^1(\Omega)$, $Q_h \subset L_*^2(\Omega)$ *provide a* ***stable finite element pair*** *(→ Def. 12.3.2.7) for the Stokes problem (12.2.2.19)/(12.3.0.2) if there is a constant* $\beta > 0$ *depending only on* Ω *and the shape regularity measure (→ Def. 3.3.2.20) of* $\mathcal{M}$ *such that*

inf-sup cond.:
$$\sup_{\mathbf{w}_h \in U_h} \frac{|\mathsf{b}(\mathbf{w}_h, q_h)|}{\|\mathbf{w}_h\|_{\mathsf{a}}} \geq \beta \|q_h\|_{L^2(\Omega)} \quad \forall q_h \in Q_h \,. \tag{12.3.2.16}$$

Recall:

Theorem 12.2.2.23. Ladyzhenskaya–Babuška–Brezzi conditions (LBB-conditions)

If the following two conditions are satisfied,

(i) the bilinear form a *is* $\mathcal{N}(\mathsf{b})$*-elliptic (**ellipticity on the kernel**)*

automatic!
$$\exists \alpha > 0: \quad |\mathsf{a}(v,v)| \geq \alpha \|v\|_U^2 \quad \forall v \in \mathcal{N}(\mathsf{b}) \,, \tag{LBB1}$$

(ii) the bilinear form b *satisfies the **inf-sup condtion***

$$\exists \beta > 0: \quad \sup_{v \in U} \frac{|\mathsf{b}(v,q)|}{\|v\|_U} \geq \beta \|q\|_Q \quad \forall q \in Q \,, \tag{LBB2}$$

then the variational saddle point problem (12.2.2.50) possesses a unique solution $(v,p) \in U \times Q$*, which satisfies*

$$\|v\|_U + \|p\|_Q \leq C \left(\sup_{w \in U} \frac{|\ell(w)|}{\|w\|_U} + \cancel{\sup_{w \in U} \frac{|g(q)|}{\|q\|_Q}} \right) , \quad g = 0 \tag{12.2.2.24}$$

with $C > 0$ *independent of* ℓ *and* g.

▷ apply to discrete SPP: $U \to U_h$, $Q \to Q_h$

▷ Solve review questions!