ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 12.3.4: The Taylor-Hood Finite Element Method

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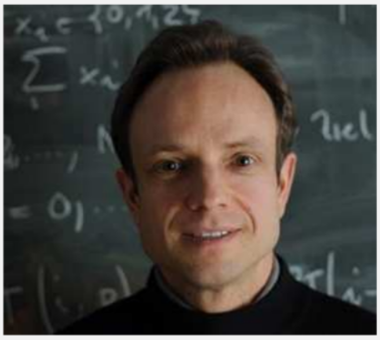


Dependencies. [Lecture → Section 12.3.3], [Lecture → Section 2.6], [Lecture → Section 3.3.5]

Duration: 30 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XII. Finite Elements for the Stokes Equations

12.3. Galerkin Discretization of the Stokes Saddle Point Problem

$v \in (H_0^1(\Omega))^d$, $p \in L_*^2(\Omega)$ s.t.

$$\begin{aligned} \int_\Omega \mu \mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{w}\,d\mathbf{x} + \int_\Omega \operatorname{div}\mathbf{w}\,p\,d\mathbf{x} &= \int_\Omega \mathbf{f}\cdot\mathbf{w}\,d\mathbf{x} \quad \forall \mathbf{w} \in (H_0^1(\Omega))^d, \\ \int_\Omega \operatorname{div}\mathbf{v}\,q\,d\mathbf{x} &= 0 \quad \forall q \in L_*^2(\Omega). \end{aligned} \tag{12.2.2.19}$$

$$\Updownarrow$$

$$\begin{array}{l} \mathbf{v} \in U := \left(H_0^1(\Omega)\right)^d \\ p \in Q := L_*^2(\Omega) \end{array} : \quad \begin{aligned} \mathsf{a}(\mathbf{v},\mathbf{w}) + \mathsf{b}(\mathbf{w},p) &= \ell(\mathbf{w}) \quad \forall \mathbf{w} \in U, \\ \mathsf{b}(\mathbf{v},q) &= 0 \quad \forall q \in Q. \end{aligned} \tag{12.3.0.2}$$

12.3.1. Pressure Instability

12.3.2. Stable Galerkin Discretization of Stokes Saddle Point Problem

12.3.3. Convergence of Stable FEM for Stokes

12.3.4 The Taylor-Hood FEM

Theorem 12.3.3.13. Convergence of stable FE for Stokes problem

If U_h, Q_h is a stable finite element pair (→ Def. 12.3.2.7) for the Stokes variational saddle point problem (12.2.2.19), *then the corresponding finite element Galerkin solution $(\mathbf{v}_h, p_h)$ satisfies*

$$\|\mathbf{v}-\mathbf{v}_h\|_{H^1(\Omega)} + \|p-p_h\|_{L^2(\Omega)} \leq C\left(\inf_{\mathbf{w}_h\in U_h}\|\mathbf{v}-\mathbf{w}_h\|_{H^1(\Omega)} + \inf_{q_h\in Q_h}\|p-q_h\|_{L^2(\Omega)}\right),$$

with a constant $C>0$ that depends only on Ω, μ, and the shape regularity of the finite element mesh.

"balanced": Should have the same order for smooth v, p

Stokes FEM, requirements

1. The pair (U_h, Q_h) of finite element spaces must be **stable** (→ Def. 12.3.2.7)
2. The velocity finite element space U_h should provide the *same rate of algebraic convergence* of the $H^1(\Omega)$-best approximation error w.r.t. $h_{\mathcal{M}} \to 0$ as the pressure space Q_h in $L^2(\Omega)$.
3. The velocity finite element space U_h should guarantee 1 and 2 with as few degrees of freedom as possible.

(2) & (3) $\Leftrightarrow$ efficiency

The **Taylor-Hood (P2-P1)** finite element method for Stokes problem relies on

- a triangular/tetrahedral or rectangular/hexahedral mesh $\mathcal{M}$ of Ω, which may even be hybrid, see Section 2.5.1,
- the velocity space: $U_h := (\mathcal{S}^0_{2,0}(\mathcal{M}))^2 \subset (H^1_0(\Omega))^d$, and
- the pressure space: $Q_h := \mathcal{S}^0_1(\mathcal{M})$, which means a *continuous pressure* approximation.

$\hat{=}$ A stable pair!

velocity: $\inf_{\mathbf{w}_h \in U_h} \|\mathbf{v} - \mathbf{w}_h\|_{H^1(\Omega)} \leq C h_{\mathcal{M}}^2 \|\mathbf{v}\|_{H^3(\Omega)}$ by Thm. 3.3.5.6,

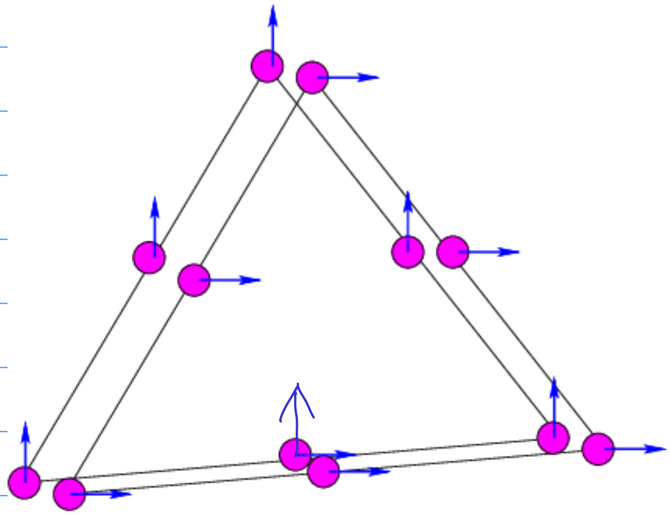
pressure: $\inf_{q_h \in \mathcal{S}_0^{-1}} \|p - q_h\|_{L^2(\Omega)} \leq C h_{\mathcal{M}}^2 \|p\|_{H^2(\Omega)}$ by Thm. 3.3.2.21.

↑ balanced

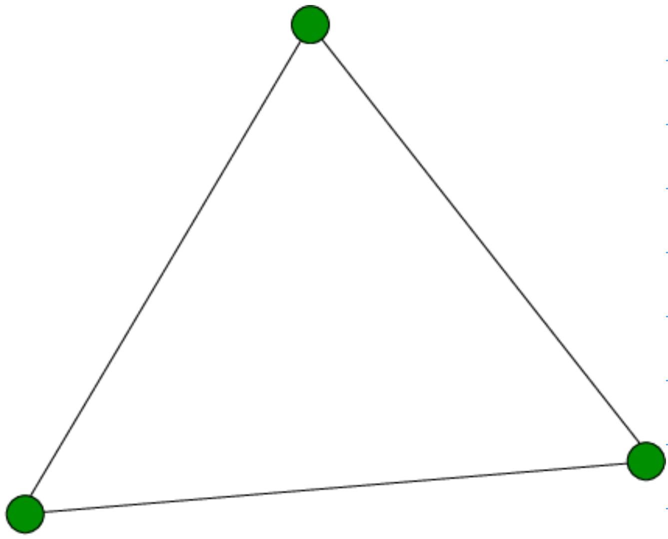
▷ $O(h_{\mathcal{M}}^2)$-cvg. of total error

U_h ≙ componentwise nodal basis function

Q ≙ tent function



sites of velocity local shape functions



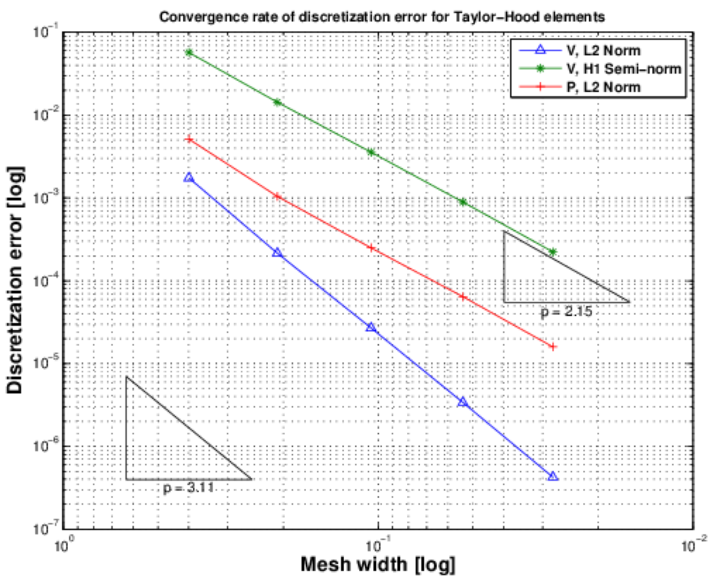
sites of pressure local shape functions

Exp. 12.3.4.5 (TH-FEM)

Monitored: Error norms $\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)}$, $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$, $\|p - p_h\|_{L^2(\Omega)}$

Observation: algebraic convergence

$$\|\mathbf{v} - \mathbf{u}_h\|_{H^1(\Omega)} = O(h_{\mathcal{M}}^2)\,,$$
$$\|\mathbf{v} - \mathbf{v}_h\|_{L^2(\Omega)} = O(h_{\mathcal{M}}^3)\,,$$
$$\|p - p_h\|_{L^2(\Omega)} = O(h_{\mathcal{M}}^2)\,.$$



▷ Look at R.Q.s