ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 13.1.1–Section 13.1.2: Fields and Integral Forms/Currents

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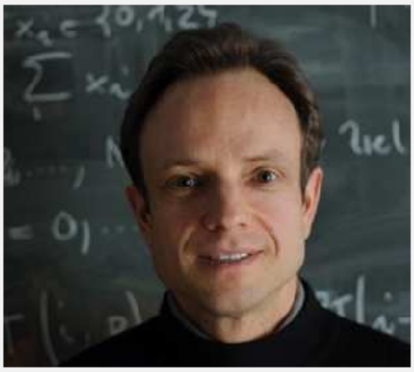


Prerequisites.

- Basic knowledge about electromagnetism as taught in first-year physics courses.
- Elementary multi-dimensional calculus

Dependencies. [Lecture → § 1.6.0.1]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XIII Finite-Element Exterior Calculus (FEEC)

13.1 (Differential) Forms

13.1.1. (EM) Fields

§ 13.1.1.1 Electric field $\underline{e}$

Std. view $\underline{e} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$

Measurement:

- local @ $x \in \Omega$: $\delta W = q\, \underline{e}(x) \cdot \delta x$

  $\uparrow$ virtual work

- practical:

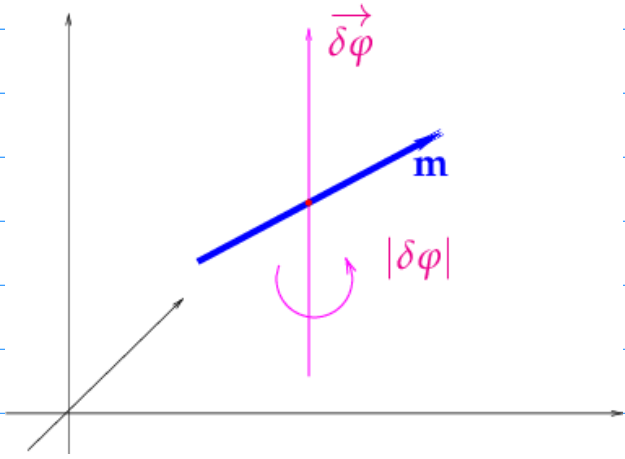
  $$W = q \int_\gamma \underline{e}(x) \cdot d\vec{s}\,, \quad \gamma = \text{directed curve}$$

$\mathbf{e}$ is a quantity that can be (path-)integrated along directed curves.

§ 13.1.1.4 Magnetic induction field $\underline{b}$

Measurement:

· local:

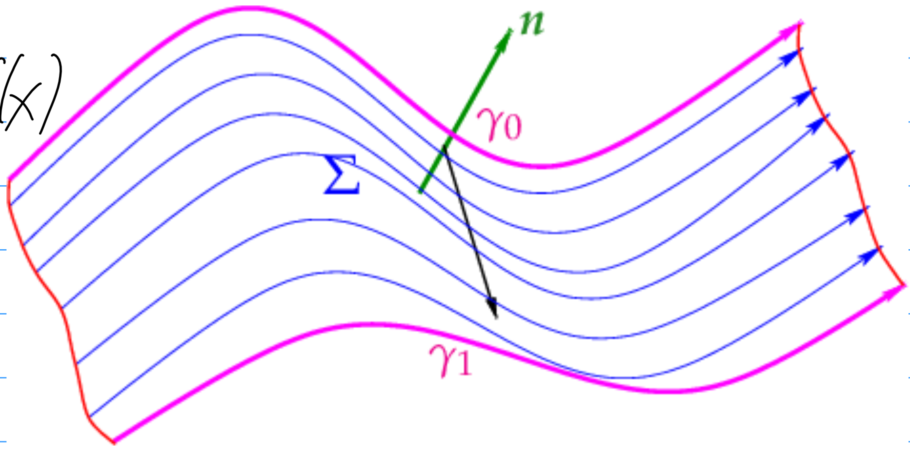


$$\delta W = b(x) \cdot (\underline{m} \times \overrightarrow{\delta\varphi})$$

magnetic dipole

· Practical: move a current carrying wire

$$W = I \int_{\Sigma} b(x) \cdot n(x) \, dS(x)$$

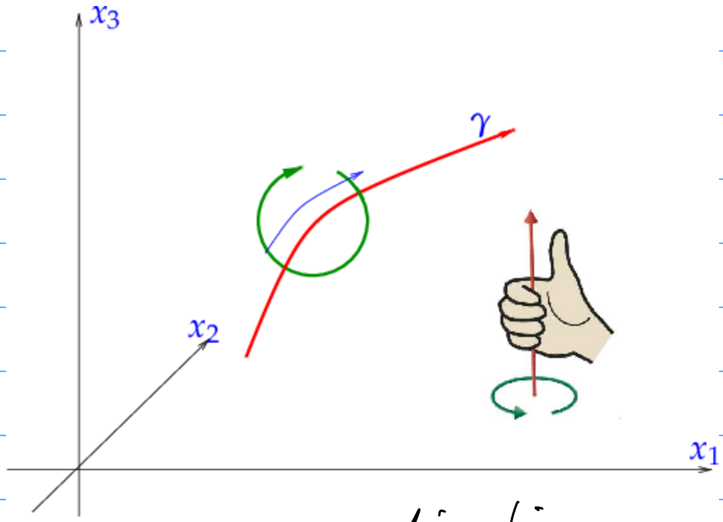


b is a quantity that assigns a total flux to oriented compact surfaces.

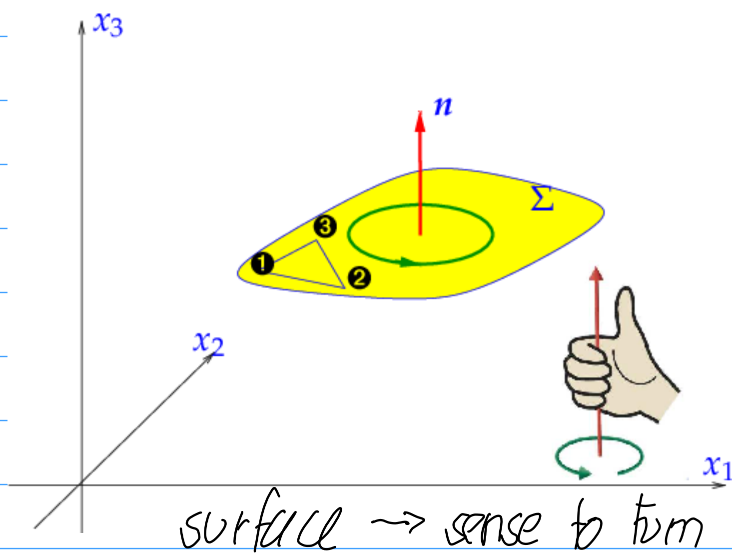
▷ $\underline{b}$, $\underline{e}$ have a totally different physical nature

13.1.2. Integral Forms

§ 13.1.2.1/2 Orientation

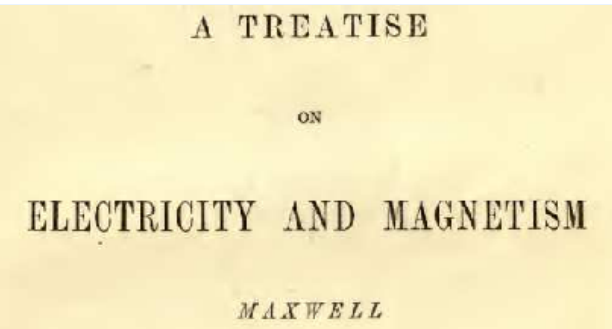


curve → direction



surface → sense to turn

$\Omega \subset \mathbb{R}^d$: $\mathcal{S}_\ell(\Omega) \hat{=}$ set of ℓ-dimensional "surfaces" (sub-manifolds), oriented


A TREATISE

ON

ELECTRICITY AND MAGNETISM

MAXWELL

Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference with respect to a line, while in the other the quantity is defined with reference to an area.

ambient domain

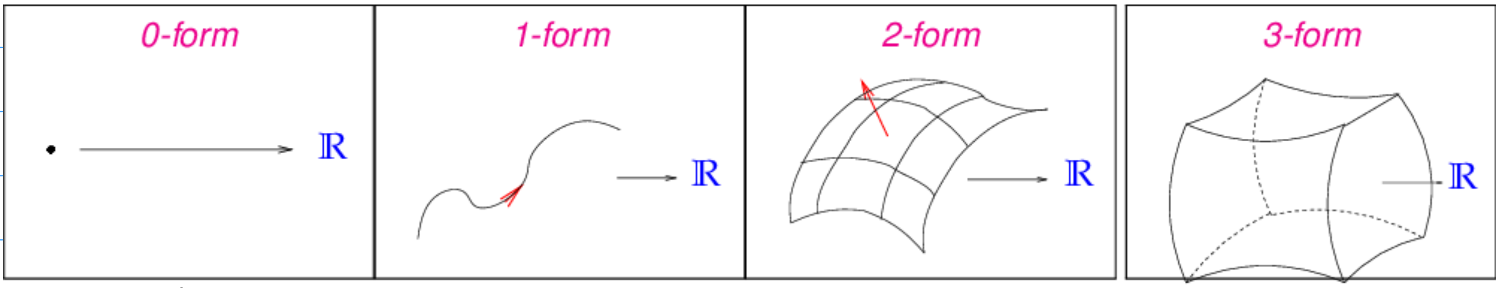
Notion 13.1.2.7. Integral ℓ-forms (IF)/ℓ-currents

An **(integral) ℓ-form/ℓ-current** $\underline{\omega}$, $0 \leq \ell \leq n$, on Σ is a continuous $(*)$ and additive $(**)$ mapping $\underline{\omega} : \bigcup_{k \leq \ell} \mathcal{S}_k(\Sigma) \to \mathbb{R}$ satisfying

$$\begin{aligned} \underline{\omega}(-\Gamma) &= -\underline{\omega}(\Gamma) \quad \forall \Gamma \in \mathcal{S}_\ell(\Sigma)\,, \\ \underline{\omega}(\gamma) &= 0 \qquad\quad\; \forall \gamma \in \mathcal{S}_k(\Sigma),\ k < \ell\,, \end{aligned} \tag{13.1.2.8}$$

where $-\Gamma$ denotes the ℓ-dimensional surface/sub-manifold with flipped orientation.

On domain $\Omega \subset \mathbb{R}^3$:



C^0-function

Notation: $I^\ell(\Omega) \triangleq$ vector space of ℓ-forms

$$\omega \in I^\ell(\Omega): \quad \langle \omega, T \rangle := \omega(T)$$

EM fields:

$\underline{e} \longleftrightarrow$ 1-form

$\underline{b} \longleftrightarrow$ 2-form

Other fields:

heat flux $\longleftrightarrow$ 2-form

mass density $\longleftrightarrow$ 3-form

temperature $\longleftrightarrow$ 0-form