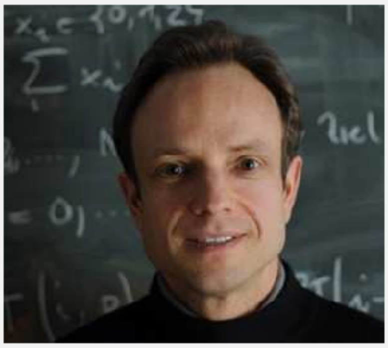
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 13.1.3: Differential Forms

Prof. R. Hiptmair, SAM, ETH Zurich

Date: May 3, 2025



Prerequisites.

- Multi-linear algebra, determinants
- Fundamentals of vector analysis

Dependencies. [Lecture → Section 13.1.2]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XIII Finite-Element Exterior Calculus (FEEC)

13.1 (Differential) Forms

13.1.3. Differential Forms

EM fields: $\underline{e} \longleftrightarrow$ 1-form, $\underline{b} \Leftrightarrow$ 2-form

Recall local measurements at $x \in \Omega$

$\underline{e}$: $\delta W(\delta x) = q\,\underline{e}(x) \cdot \delta x$

$\underline{b}$: $\delta W(\underline{m}, \delta\vec{\varphi}) = \underline{b}(x) \cdot (\underline{m} \times \delta\vec{\varphi})$

$\mathbf{e}(\boldsymbol{x}, t)$ can be regarded as a linear mapping from displacements into $\mathbb{R}$.

$\mathbf{b}(\boldsymbol{x}, t)$ should be read as an anti-symmetric (*) bilinear form $(\overrightarrow{\delta\varphi}, \mathbf{m}) \to \mathbb{R}$.

Definition 13.1.3.4. Alternating ℓ-linear form

An **alternating ℓ-linear form** μ, $\ell \in \mathbb{N}$, on a real vector space V is a mapping

$$\mu : \underbrace{V \times V \times \cdots \times V}_{\ell \text{ times}} \to \mathbb{R}\,,$$

which (i) is *linear* in *each* of its ℓ arguments, and which
(ii) changes sign, when any two arguments are swapped ("alternating").

$\Leftrightarrow$ $=0$ when any two arg. vectors are the same

$\Leftrightarrow$ $=0$ when args. are lin. dep.

Notation: $\Lambda^\ell(V) \hat{=}$ vector space

$\Lambda^0(V) = \mathbb{R}$

$\Lambda^1(V) \hat{=}$ linear forms on V

$n := \dim V$: $\dim \Lambda^\ell(V) = \binom{n}{\ell}$ $\hat{=}$ # ℓ-subsets of a basis of V

$0 \leq \ell \leq n$

Abstract: quantities that can be measured locally

mapping $\Omega \subset \mathbb{R}^d \longrightarrow \Lambda^\ell(\mathbb{R}^d)$

$\hookrightarrow$ = differential ℓ-form on Ω

$\rightarrow$ Vector space $\Lambda^\ell(\Omega)$

$\{b_1, \dots, b_{\binom{d}{\ell}}\} \hat{=}$ basis of $\Lambda^\ell(\mathbb{R}^d)$

$\omega \in \Lambda^\ell(\Omega)$: $\omega(\boldsymbol{x}) = \sum_{j=1}^{\binom{d}{\ell}} u_j(\boldsymbol{x})\beta_j \quad \Rightarrow \quad \Lambda^\ell(\Omega) \cong \left\{\boldsymbol{x} \in \Omega \mapsto \mathbb{R}^{\binom{d}{\ell}}\right\}.$

$\hookrightarrow$ coefficient functions

▷ Function spaces of DF

$$X\Lambda^\ell(\Omega) := \left\{\omega \in \Lambda^\ell(\Omega) \text{ with all coefficient functions } \in X(\Omega)\right\}.$$

e.g. : $C^0\Lambda^\ell(\Omega)$, $L^2\Lambda^\ell(\Omega)$

$d=3$: "Canonical" choice of basis

	basis (set of alternating ℓ-linear forms on $\mathbb{R}^3$)
$\ell=0$	$\{1\}$
$\ell=1$	$\{\{\mathbf{v}\mapsto v_1\},\{\mathbf{v}\mapsto v_2\},\{\mathbf{x}\mapsto v_3\}\}$
$\ell=2$	$\{\{(\mathbf{v},\mathbf{w})\mapsto v_2w_3-v_3w_2\},\{(\mathbf{v},\mathbf{w})\mapsto v_3w_1-v_1w_3\},\{(\mathbf{v},\mathbf{w})\mapsto v_1w_2-v_2w_1\}\}$
$\ell=3$	$\{(\mathbf{v},\mathbf{w},\mathbf{q})\mapsto \det(\mathbf{v},\mathbf{w},\mathbf{q})\}$

Table 13.1.3.15: Standard choices of bases for $\Lambda^\ell(\mathbb{R}^3)$, $\mathbf{v},\mathbf{w},\mathbf{q}\in\mathbb{R}^3$

vector/function proxies

ℓ	Differential ℓ-form	Related function u/vectorfield $\mathbf{u}$	
0	$\mathbf{x}\mapsto\omega(\mathbf{x})$	$u(\mathbf{x}):=\omega(\mathbf{x})$	$u:\Omega\to\mathbb{R}$
1	$\mathbf{x}\mapsto\{\mathbf{v}\mapsto\omega(\mathbf{x})(\mathbf{v})\}$	$\mathbf{u}(\mathbf{x})\cdot\mathbf{v}:=\omega(\mathbf{x})(\mathbf{v})$	$\mathbf{u}:\Omega\to\mathbb{R}^3$
2	$\mathbf{x}\mapsto\{(\mathbf{v}_1,\mathbf{v}_2)\mapsto\omega(\mathbf{x})(\mathbf{v}_1,\mathbf{v}_2)\}$	$\mathbf{u}(\mathbf{x})\cdot(\mathbf{v}_1\times\mathbf{v}_2):=\omega(\mathbf{x})(\mathbf{v}_1,\mathbf{v}_2)$	$\mathbf{u}:\Omega\to\mathbb{R}^3$
3	$\mathbf{x}\mapsto\{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)\mapsto\omega(\mathbf{x})(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)\}$	$u(\mathbf{x})\det(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3):=\omega(\mathbf{x})(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$	$u:\Omega\to\mathbb{R}$

Table 13.1.3.19: "Canonical" identification of differential forms with functions/vector fields in $\mathbb{R}^3$

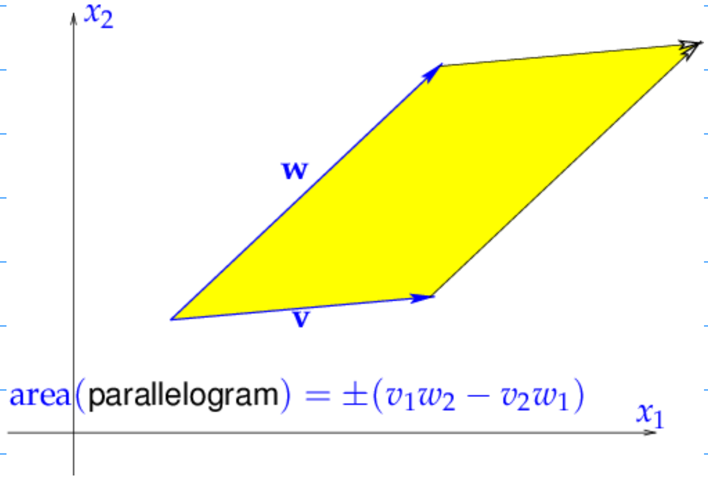
Notation : v.p.(ω)

$d=2$:

ℓ	Differential ℓ-form	Related function u/vectorfield $\mathbf{u}$	
0	$\mathbf{x}\mapsto\omega(\mathbf{x})$	$u(\mathbf{x}):=\omega(\mathbf{x})$	$u:\Omega\to\mathbb{R}$
1	$\mathbf{x}\mapsto\{\mathbf{v}\mapsto\omega(\mathbf{x})(\mathbf{v})\}$	$\left\{\begin{array}{ll}\text{Option I:} & \mathbf{u}(x)\cdot\mathbf{v}\\ \text{Option II:} & \mathbf{u}(\mathbf{x})^\perp\cdot\mathbf{v}\end{array}\right. := \omega(\mathbf{x})(\mathbf{v})$	$\mathbf{u}:\Omega\to\mathbb{R}^2$
2	$\mathbf{x}\mapsto\{(\mathbf{v}_1,\mathbf{v}_2)\mapsto\omega(\mathbf{x})(\mathbf{v}_1,\mathbf{v}_2)\}$	$u(\mathbf{x})\det(\mathbf{v}_1,\mathbf{v}_2):=\omega(\mathbf{x})(\mathbf{v}_1,\mathbf{v}_2)$	$u:\Omega\to\mathbb{R}$

Table 13.1.3.23: 2D Euclidean function/vector proxies for differential forms on $\Omega\subset\mathbb{R}^2$

Next: DF $\xrightarrow{\text{integration}}$ IF

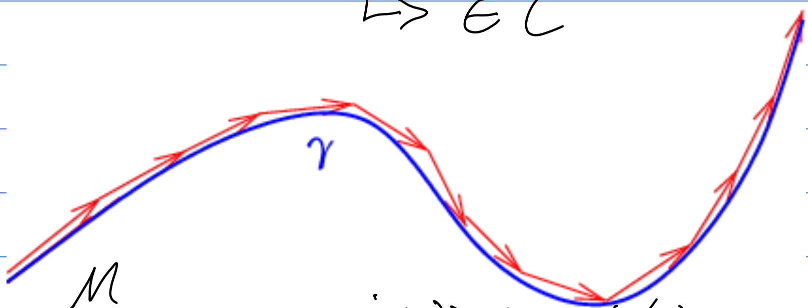


$\omega \in \Lambda^2(\mathbb{R}^2)$

$\omega(v,w) \triangleq$ signed area of parallelogram

$l=1$: $\gamma : [0,1] \longrightarrow \Omega \subset \mathbb{R}^d \triangleq$ directed curve

$\hookrightarrow \in C^2$



Σ contributions of oriented line segments

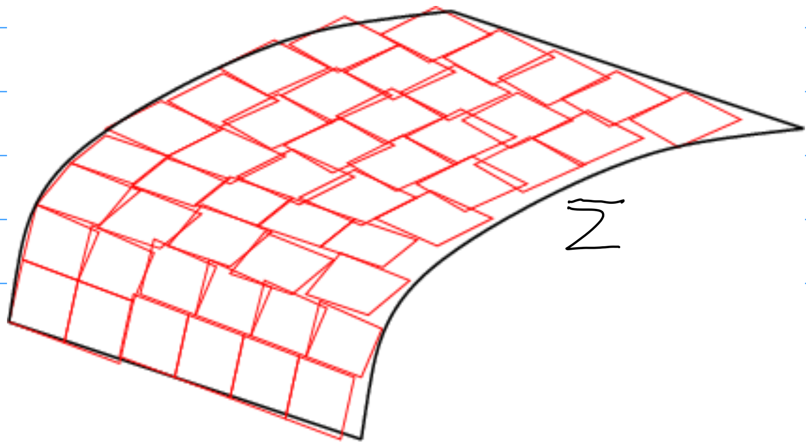
$$\sum_{i=1}^{M} \omega(\gamma(i/M))(\gamma(i/M) - \gamma(\tfrac{i-1}{M})) = \quad \text{Taylor exp.}$$

$$\sum_i \omega(\ldots)(\dot{\gamma}(i/M)\tfrac{1}{M} + O(M^{-2}))$$

$$\xrightarrow[\text{"Riemann sum"}]{M\to\infty} \int_0^1 \omega(\gamma(s))(\dot{\gamma}(s))\,ds$$

$$[\,\omega \in C^0\Lambda^1(\Omega)\,]$$

$l=2$:



Σ

General: $\Gamma = \Psi(\Pi)$, $\Pi \subset \mathbb{R}^\ell$
[parameterization] [parameter domain]

$$\text{“}\langle \underline{\omega}, \Gamma\rangle\text{”} = \int_\Gamma \omega := \int_\Pi \omega(\Psi(p))\underbrace{\left((D\Psi)(p))_{:,1}, \dots, (D\Psi)(p))_{:,\ell}\right)}_{\text{tangent vector}} \, dp \,. \tag{13.1.3.28}$$

$\uparrow$
interpretation as integral form

$\triangleright$ Embedding: $C^0\Lambda^\ell(\Omega) \subset \mathcal{I}^\ell(\Omega)$

§ 13.1.3.30: Integration of 3D vector proxies

$\ell = 1$: $\displaystyle\int_\gamma \omega = \int_0^1 \underbrace{\omega(\gamma(s))(\dot{\gamma}(s))}_{u(\gamma(s))\cdot\dot{\gamma}(s)}\,ds = \int_\gamma u(x)\cdot d\vec{s}(x)$

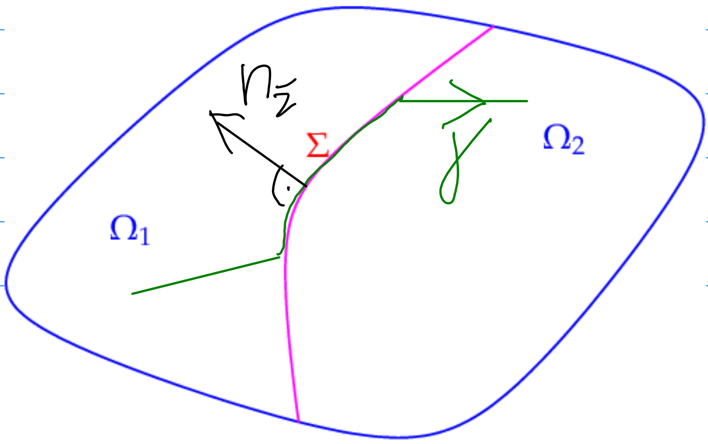
$u := \text{v.p.}(\omega)$

a path integral

$$\int_\Gamma \omega = \begin{cases} \omega(x) & \text{for } \ell = 0,\quad \Gamma = \{x\},\ x \in \Omega\,, \\ \int_\Gamma \text{v.p.}(\omega)(x)\cdot d\vec{s}(x) & \text{for } \ell = 1\,, \\ \int_\Gamma \text{v.p.}(\omega)(x)\cdot n(x)\,dS(x) & \text{for } \ell = 2\,, \\ \int_\Gamma \text{v.p.}(\omega)(x)\,dx & \text{for } \ell = 3\,. \end{cases} \tag{13.1.3.34}$$

flux integral

Discontinuous DF, still an IF



$\omega_i \in C\Lambda^1(\overline{\Omega_i})$

$$\omega := \begin{cases} \omega_1 & \text{in } \Omega_1 \\ \omega_2 & \text{in } \Omega_2 \end{cases}$$

$\int_\gamma \omega$ unique, if $\omega_{1|\Sigma}$ and $\omega_{2|\Sigma}$ agree on the tangent space of Σ

Continuous differential ℓ-forms $\omega_1 \in C^0\Lambda^\ell(\overline{\Omega}_1)$, $\omega_2 \in C^0\Lambda^\ell(\overline{\Omega}_2)$, give rise to a valid integral ℓ-form on Ω by piecewise integration, if and only if

$$\omega_1(\boldsymbol{x})(\mathbf{t}_1,\dots,\mathbf{t}_\ell) = \omega_2(\boldsymbol{x})(\mathbf{t}_1,\dots,\mathbf{t}_\ell) \quad \forall \mathbf{t}_1,\dots,\mathbf{t}_\ell \in \mathcal{T}_{\boldsymbol{x}}(\Sigma) \quad \forall \boldsymbol{x} \in \Sigma\,. \tag{13.1.3.37}$$

▷ (Partial) continuity requirement for 3D VP:

$\ell = 0$: C^0-continuity

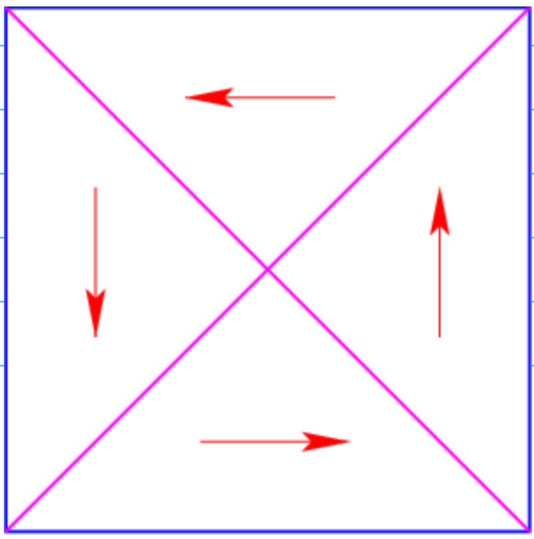
$\ell = 1$: tangential cont. $[\mathrm{v.p.}(\omega) \times n_\Sigma]_\Sigma = 0$

$\ell = 2$: normal cont. $[\mathrm{v.p.}(\omega) \cdot n_\Sigma]_\Sigma = 0$

$\ell = 3$: /

Compare EM text books: @ material interfaces Σ

$[b \cdot n_\Sigma]_\Sigma = 0$ $\qquad$ $[e \times n_\Sigma]_\Sigma = 0$



2D vector field, p.w. constant normally continuous

§ 13.1.3.38 Zero b.c. for IF/DF on $\partial\Omega$

$\omega \in \mathcal{I}^{\ell}(\Omega)$:

ω vanishes on $\partial\Omega$ $\Longleftrightarrow$ $\omega(F) = 0$ $\forall F \in \mathcal{S}_{\ell}(\Omega)$, $F \subset \partial\Omega$

DF : $\omega \in C^0\Lambda^{\ell}(\overline{\Omega})$

ω vanishes on $\Sigma := \partial\Omega$ $\Leftrightarrow$

$$\omega(\boldsymbol{x})(\mathbf{t}_1,\ldots,\mathbf{t}_\ell) = 0 \quad \forall \text{tangent vectors } \mathbf{t}_1,\ldots,\mathbf{t}_\ell \in \mathcal{T}_{\boldsymbol{x}}(\Sigma),\ \forall \boldsymbol{x} \in \Sigma\,. \tag{13.1.3.41}$$

[zero tangential b.c.]

For 3D VP :

$\ell = 1$: $\quad$ v.p. $(\omega) \times n(x) = 0 \quad \forall x \in \partial\Omega$

$\ell = 2$: $\quad$ v.p. $(\omega)(x) \cdot n(x) = 0 \quad \forall x \in \partial\Omega$

EM :

$$\check{\mathbf{e}}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}) = \mathbf{0} \quad \forall \boldsymbol{x} \in \Sigma \quad \hat{=} \text{ perfectly electrically conducting (**PEC**) b.c.} \tag{13.1.3.43}$$

$$\check{\mathbf{b}}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0 \quad \forall \boldsymbol{x} \in \Sigma \quad \hat{=} \text{ zero flux b.c.} \tag{13.1.3.44}$$

$$\mathrm{DF} \underset{\text{localization (?)}}{\overset{\text{integration}}{\rightleftarrows}} \mathrm{IF}$$

§ 13.1.3.44

Why alternating ?

$$\omega(\boldsymbol{x})(\mathbf{v}_1,\dots,\mathbf{v}_\ell) := \lim_{t\to 0}\frac{1}{t^\ell}\langle \underline{\omega}, P_t(\boldsymbol{x})\rangle \,, \tag{13.1.3.46}$$

$$P_t(\boldsymbol{x}) := \left\{ \boldsymbol{x} + t\sum_{i=1}^{\ell} \xi_i \mathbf{v}_i : \ 0 \le \xi_i \le 1,\ i = 1,\dots,\ell \right\}, \qquad \mathbf{v}_i \in \mathbb{R}^d$$

Existence of limit will depend on "smoothness" of $\underline{\omega}$