ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 13.1.4: Exterior Calculus

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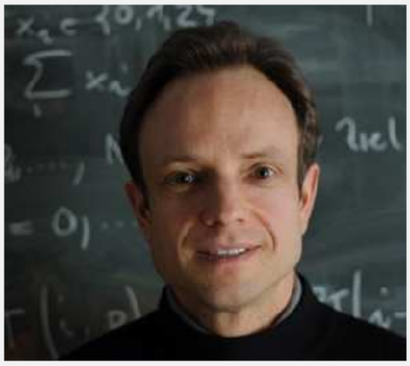
Date: May 3, 2025



Prerequisites.

- Multi-dimensional calculus and vector analysis, [Lecture → Section 0.3.2.2], [Lecture → Section 0.3.2.5]

Dependencies. [Lecture → Section 13.1.2], [Lecture → Section 13.1.3], [Lecture → Section 1.5.2], [Lecture → Section 2.8.3]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XIII Finite-Element Exterior Calculus (FEEC)

13.1 (Differential) Forms

13.1.4. Exterior Calculus

13.1.4.1 Pullback of integral/differential forms

- $\Phi : \hat{\Omega} \to \Omega$, $\hat{\Omega}, \Omega$ domains
  - ↳ C^1 & bijective
- $\underline{\omega} \in \mathcal{I}^\ell(\Omega)$

$$\langle \Phi^* \underline{\omega}, \Gamma \rangle := \langle \underline{\omega}, \Phi(\Gamma) \rangle \quad \forall \Gamma \in \mathcal{S}_\ell(\hat{\Omega})$$

↳ pullback : "$\langle \Phi^* \cdot, \cdot \rangle = \langle \cdot, \Phi \cdot \rangle$"

↳ linear operator $\mathcal{I}^\ell(\Omega) \to \mathcal{I}^\ell(\hat{\Omega})$

$\ell = 0$: $\quad \Phi^* \underline{\omega}(x) = \underline{\omega}(\Phi(x))$

For DF :

$$\begin{array}{ccc} \text{differential } \ell\text{-form on } \Sigma & \xrightarrow{\text{integration according to (13.1.3.28)}} & \text{integral } \ell\text{-form on } \Sigma \\ \downarrow ? & & \downarrow \Phi^* \\ \text{differential } \ell\text{-form on } \widehat{\Sigma} & \xleftarrow{\text{localization according to (13.1.3.46)}} & \text{integral } \ell\text{-form on } \widehat{\Sigma} \end{array} \quad (13.1.4.3)$$

Definition 13.1.4.4. Pullback/transformation for differential forms

Given a C^1-mapping $\mathbf{\Phi} : \widehat{\Omega} \to \Omega$, $\widehat{\Omega}, \Omega \subset \mathbb{R}^d$ domains, the **pullback** $\mathbf{\Phi}^* : \Lambda^\ell(\Omega) \to \Lambda^\ell(\widehat{\Omega})$, $0 \le \ell \le d$, is defined as [local !]

$$(\mathbf{\Phi}^*\omega)(\widehat{x})(\mathbf{v}_1, \dots, \mathbf{v}_\ell) = \omega(\mathbf{\Phi}(\widehat{x}))(\mathrm{D}\mathbf{\Phi}(\widehat{x})\mathbf{v}_1, \dots, \mathrm{D}\mathbf{\Phi}(\widehat{x})\mathbf{v}_\ell) \quad \forall \mathbf{v}_i \in \mathbb{R}^d \quad \forall \widehat{x} \in \widehat{\Omega}\,.$$

Theorem 13.1.4.5. Invariance of integrals under pullback

Using the notations of Def. 13.1.4.4, the pullback $\mathbf{\Phi}^* : \Lambda^\ell(\Omega) \to \Lambda^\ell(\widehat{\Omega})$ *satisfies*

$$\int_{\widehat{\Gamma}} (\mathbf{\Phi}^*\omega) = \int_{\mathbf{\Phi}(\widehat{\Gamma})} \omega \qquad \forall \widehat{\Gamma} \in \mathcal{S}_\ell(\widehat{\Omega})\,, \quad (13.1.4.6)$$

which ensures compatibility with the pullback of integral forms.

For 3D VP : $\quad x := \Phi(\widehat{x})$

forms/vector proxies		Pullback induced by $\mathbf{\Phi} : \widehat{\Omega} \to \Omega$
$\ell = 0$	$u := \mathrm{v.p.}(\omega)$, $\widehat{u} := \mathrm{v.p.}(\mathbf{\Phi}^*\omega)$	$\widehat{u}(\widehat{x}) = u(x)$
$\ell = 1$	$\mathbf{u} := \mathrm{v.p.}(\omega)$, $\widehat{\mathbf{u}} := \mathrm{v.p.}(\mathbf{\Phi}^*\omega)$	$\widehat{\mathbf{u}}(\widehat{x}) = \mathrm{D}\mathbf{\Phi}(\widehat{x})^\top \mathbf{u}(x)$
$\ell = 2$	$\mathbf{u} := \mathrm{v.p.}(\omega)$, $\widehat{\mathbf{u}} := \mathrm{v.p.}(\mathbf{\Phi}^*\omega)$	$\widehat{\mathbf{u}}(\widehat{x}) = \det \mathrm{D}\mathbf{\Phi}(\widehat{x}) \mathrm{D}\mathbf{\Phi}(\widehat{x})^{-1}\mathbf{u}(x)$
$\ell = 3$	$u := \mathrm{v.p.}(\omega)$, $\widehat{u} := \mathrm{v.p.}(\mathbf{\Phi}^*\omega)$	$\widehat{u}(\widehat{x}) = \det \mathrm{D}\mathbf{\Phi}(\widehat{x}) u(x)$

Table 13.1.4.13: Pullback of differential forms in terms of *3D* Euclidean vector proxies

13.1.4.2 Exterior Derivative

§ 13.1.4.15 Maxwell's equation

EM quant.

- the electric field $\mathbf{e}$ (SI units $[\mathbf{e}] = \mathrm{V\,m^{-1}}$), see § 13.1.1.1,
- the magnetic induction field $\mathbf{b}$ (SI units $[\mathbf{b}] = \mathrm{V\,s\,m^{-2}}$), see § 13.1.1.4.
- the **magnetic field** $\mathbf{h}$ (SI units $[\mathbf{h}] = \mathrm{A\,m^{-1}}$),
- the **electric displacement field** $\mathbf{d}$ (SI units $[\mathbf{d}] = \mathrm{A\,s\,m^{-2}}$), and
- the **electric current** $\mathbf{j}$ (SI units $[\mathbf{j}] = \mathrm{A\,m^{-2}}$).

All vectorfields $\Omega \longrightarrow \mathbb{R}^3$?

e.g. $\int_\Sigma \mathbf{j} \cdot \mathbf{n}\, dS$ = total current through Σ

electric field $\mathbf{e}$	$\leftrightarrow$	(integral)	1-form	$\underline{\mathbf{e}}$	magnetic field $\mathbf{h}$	$\leftrightarrow$	(integral)	1-form	$\underline{\mathbf{h}}$
magnetic induction $\mathbf{b}$	$\leftrightarrow$	(integral)	2-form	$\underline{\mathbf{b}}$	electric displacement $\mathbf{d}$	$\leftrightarrow$	(integral)	2-form	$\underline{\mathbf{d}}$
					electric current $\mathbf{j}$	$\leftrightarrow$	(integral)	2-form	$\underline{\mathbf{j}}$

Table 13.1.4.16: Integral-form models for the four basic electrodynamic quantities

Maxwell's equations

$\forall\, t \longmapsto \Sigma(t) \in S_2(\Omega)$

Faraday's law: $$\langle \underline{\mathbf{e}}(t), \partial\Sigma(t)\rangle = -\frac{d}{dt}\langle \underline{\mathbf{b}}(t), \Sigma(t)\rangle\,, \qquad (13.1.4.18)$$

Ampere's law: $$\langle \underline{\mathbf{h}}(t), \partial\Sigma(t)\rangle = \frac{d}{dt}\langle \underline{\mathbf{d}}(t), \Sigma(t)\rangle + \left\langle \underline{\mathbf{j}}(t), \Sigma(t)\right\rangle\,. \qquad (13.1.4.19)$$

$\partial : S_2(\Omega) \longrightarrow S_1(\Omega)$: boundary operator

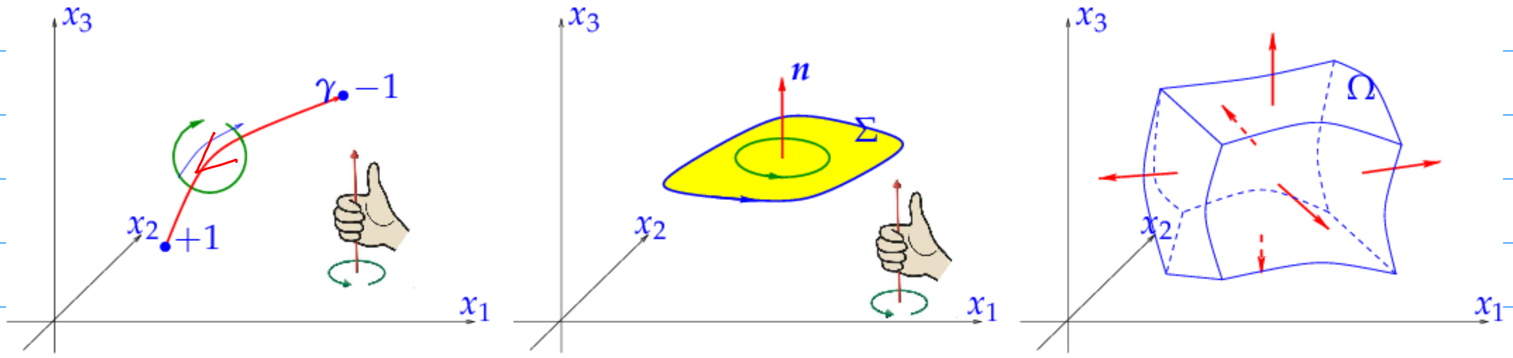
DF:

Faraday's law (13.1.4.18): $$\int_{\partial\Sigma(t)} \mathbf{e}(t) = -\frac{d}{dt}\int_{\Sigma(t)} \mathbf{b}(t)\,, \qquad (13.1.4.21)$$

Ampere's law (13.1.4.19): $$\int_{\partial\Sigma(t)} \mathbf{h}(t) = \frac{d}{dt}\int_{\Sigma(t)} \mathbf{d}(t) + \int_{\Sigma(t)} \mathbf{j}(t)\,. \qquad (13.1.4.22)$$

Required : orientation of $\partial\Sigma(t)$
= induced orientation

In 3D:



Definition 13.1.4.27. Exterior derivative of integral forms

Let Σ be an n-dimensional manifold/domain, $n \in \mathbb{N}$. For $0 \leq \ell < n$ the **exterior derivative** operators (for integral forms)

$$\underline{\mathrm{d}}_\ell : \mathcal{I}^\ell(\Sigma) \to \mathcal{I}^{\ell+1}(\Sigma)$$

are defined as

$$\langle \underline{\mathrm{d}}_\ell \underline{\omega}, \Gamma \rangle = \langle \underline{\omega}, \partial\Gamma \rangle \quad \forall \underline{\omega} \in \mathcal{I}^\ell(\Sigma) \quad \forall \Gamma \in \mathcal{S}_{\ell+1}(\Sigma) \,, \tag{13.1.4.28}$$

where the boundary $\partial\Gamma$ is endowed with the induced orientation.

" $\langle \underline{d}_\ell \cdot , \cdot \rangle = \langle \cdot , \partial \cdot \rangle$ "

Faraday's law: $$\langle \underline{\mathbf{e}}(t), \partial\Sigma(t) \rangle = -\frac{d}{dt} \langle \underline{\mathbf{b}}(t), \Sigma(t) \rangle \,, \tag{13.1.4.18}$$

Ampere's law: $$\langle \underline{\mathbf{h}}(t), \partial\Sigma(t) \rangle = \frac{d}{dt} \langle \underline{\mathbf{d}}(t), \Sigma(t) \rangle + \left\langle \underline{\mathbf{j}}(t), \Sigma(t) \right\rangle \,. \tag{13.1.4.19}$$

as

$$\underline{\mathrm{d}}_1 \underline{\mathbf{e}}(t) = -\frac{d}{dt} \underline{\mathbf{b}}(t) \quad , \quad \underline{\mathrm{d}}_1 \underline{\mathbf{h}}(t) = \frac{d}{dt} \underline{\mathbf{d}}(t) + \underline{\mathbf{j}}(t) \,. \tag{13.1.4.30}$$

"Obvious": $\partial\partial\Gamma = \emptyset$

$\Rightarrow \quad \mathsf{d}_{\ell+1} \circ \mathsf{d}_\ell = 0$

"Obvious": $\partial\Phi(\Gamma) = \Phi(\partial\Gamma)$

$\Rightarrow \quad \Phi^* \circ \mathsf{d}_\ell = \mathsf{d}_\ell \circ \Phi^*$

§ 13.1.4.35 localization: d_ℓ for DF

$$\begin{array}{ccc} \text{differential } \ell\text{-form on } \Omega & \xrightarrow{\text{integration according to (13.1.3.28)}} & \text{integral } \ell\text{-form on } \Omega \\ ?\big\downarrow & & \big\downarrow \mathsf{d} \\ \text{differential } \ell+1\text{-form on } \Omega & \xleftarrow{\text{localization according to (13.1.3.46)}} & \text{integral } \ell+1\text{-form on } \Omega \end{array}$$

Definition 13.1.4.38. Exterior derivative for C^1 differential forms

The **exterior derivative** of a continuously differentiable differential form $\omega \in C^1\Lambda^\ell(\Omega)$ of order $\ell \in \{0,\dots,d-1\}$ on a domain $\Omega \subset \mathbb{R}^d$ is the continuous differential $\ell+1$-form $\mathsf{d}_\ell\omega$ defined as[a]

$$\mathsf{d}_\ell\omega(\boldsymbol{x})(\mathbf{v}_1,\dots,\mathbf{v}_{\ell+1}) := \sum_{k=1}^{\ell+1}(-1)^{k-1}\mathsf{D}\omega(\boldsymbol{x})\mathbf{v}_k(\mathbf{v}_1,\dots,\cancel{\mathbf{v}_k},\dots,\mathbf{v}_{\ell+1}) \qquad (13.1.4.39)$$

for all $\boldsymbol{x} \in \Omega$ and "tangent vectors" $\mathbf{v}_i \in \mathbb{R}^d$.

[a] In this formula $\mathsf{D}\omega : \Omega \to \mathcal{L}(\mathbb{R}^d, \Lambda^\ell(\mathbb{R}^d))$ is the (Fréchet) derivative of ω, which maps into the space of linear mappings $\mathbb{R}^d \to \Lambda^\ell(\mathbb{R}^d)$.

$\Rightarrow \quad \mathsf{d} : C^1\Lambda^\ell(\Omega) \longrightarrow C^0\Lambda^\ell(\Omega)$

For 3D VP:

$$\begin{array}{ccccccc} C^\infty\Lambda^0(\Omega) & \xrightarrow{\mathsf{d}_0} & C^\infty\Lambda^1(\Omega) & \xrightarrow{\mathsf{d}_1} & C^\infty\Lambda^2(\Omega) & \xrightarrow{\mathsf{d}_2} & C^\infty\Lambda^3(\Omega) \\ \text{v.p.}\big\downarrow & & \text{v.p.}\big\downarrow & & \text{v.p.}\big\downarrow & & \text{v.p.}\big\downarrow \\ C^\infty(\Omega,\mathbb{R}) & \xrightarrow{\mathbf{grad}} & C^\infty(\Omega,\mathbb{R}^3) & \xrightarrow{\mathbf{curl}} & C^\infty(\Omega,\mathbb{R}^3) & \xrightarrow{\mathrm{div}} & C^\infty(\Omega,\mathbb{R})\,. \end{array} \qquad (13.1.4.47)$$

Theorem 13.1.4.57. Generalized Stokes theorem

For every domain $\Omega \subset \mathbb{R}^d$ *and* $\ell \in \{0,\dots,d-1\}$ *holds true*

$$\int_\Gamma \mathsf{d}_\ell \omega = \int_{\partial\Gamma} \omega \quad \forall \omega \in C^1\Lambda^\ell(\Sigma)\,, \quad \forall \Gamma \in \mathcal{S}_{\ell+1}(\Omega)\,. \tag{13.1.4.58}$$

$\hat{=}$ $d_\ell = \underline{d_\ell}$ on DF

3D V.P.:

degree ℓ	function/vector proxy $u/\mathbf{u}$	$\int_\Gamma \omega$	$\mathsf{d}_\ell\omega$
0	$u(\mathbf{x}) := \omega(\mathbf{x})$	$u(\mathbf{x})$	$\mathbf{grad}\, u$
1	$\mathbf{u}(\mathbf{x})\cdot\mathbf{v} := \omega(\mathbf{x})(\mathbf{v})$	$\int_\Gamma \mathbf{u}(\mathbf{x})\cdot \mathrm{d}\vec{s}(\mathbf{x})$	$\mathbf{curl}\,\mathbf{u}$
2	$\mathbf{u}(\mathbf{x})\cdot(\mathbf{v}_1\times\mathbf{v}_2) := \omega(\mathbf{x})(\mathbf{v}_1,\mathbf{v}_2)$	$\int_\Gamma \mathbf{u}(\mathbf{x})\cdot \boldsymbol{n}(\mathbf{x})\,\mathrm{d}S(\mathbf{x})$	$\mathrm{div}\,\mathbf{u}$
3	$u(\mathbf{x})\det(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) := \omega(\mathbf{x})(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$	$\int_\Gamma u(\mathbf{x})\,\mathrm{d}\mathbf{x}$	—

Table 13.1.4.48: Integration and exterior derivative of differential forms in terms of *3D* Euclidean vector proxies; $\Gamma \hat{=} \ell$-dimensional sub-manifold of Ω, $\Gamma \in \mathcal{S}_\ell(\Omega)$

$\ell = 0$: $\displaystyle\int_\gamma \mathbf{grad}\, u(\boldsymbol{x})\cdot \mathrm{d}\vec{s}(\boldsymbol{x}) = u(\boldsymbol{p}_1) - u(\boldsymbol{p}_0)$, γ a curve from $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$, (13.1.4.61)

$\ell = 1$: $\displaystyle\int_\Gamma \mathbf{curl}\,\mathbf{u}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) = \int_{\partial\Gamma}\mathbf{u}(\boldsymbol{x})\cdot\mathrm{d}\vec{s}(\boldsymbol{x})$, Γ an oriented, compact 2-surface, (13.1.4.62)

$\ell = 2$: $\displaystyle\underbrace{\int_V \mathrm{div}\,\mathbf{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\partial V}\mathbf{u}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x})}_{\to \text{Gauss theorem}}$, V a volume. (13.1.4.63)

→ Classical Stokes thm.

→ Gauss theorem

Theorem 13.1.4.64. $\mathsf{d}\circ\mathsf{d} = 0$

For every domain $\Omega \subset \mathbb{R}^d$ *and* $\ell \in \{0,\dots,d-2\}$ *holds true*

$$\mathsf{d}_{\ell+1}\circ\mathsf{d}_\ell = 0 \quad \text{on} \quad C^2\Lambda^\ell(\Omega) \quad \Leftrightarrow \quad \mathcal{R}(\mathsf{d}_\ell) \subset \mathcal{N}(\mathsf{d}_{\ell+1})\,, \tag{13.1.4.65}$$

that is, the image of d_ℓ *lies in the kernel of* $\mathsf{d}_{\ell+1}$.

$\rightarrow$ curl $\circ$ grad $= 0$, div $\circ$ curl $= 0$

Theorem 13.1.4.70. Exterior derivatives and pullbacks commute

Let $\mathbf{\Phi}$ *be a continuously differentiable mapping* $\mathbf{\Phi} : \widehat{\Omega} \to \Omega, \widehat{\Omega}, \Omega \subset \mathbb{R}^d$ *domains. Then*

$$\mathrm{d}_\ell(\mathbf{\Phi}^*\omega) = \mathbf{\Phi}^*(\mathrm{d}_\ell\omega) \quad \forall \omega \in C^1\Lambda^\ell(\Omega) \quad \Leftrightarrow \quad \begin{array}{ccc} C^1\Lambda^\ell(\Omega) & \xrightarrow{\mathrm{d}_\ell} & C^0\Lambda^{\ell+1}(\Omega) \\ \mathbf{\Phi}^* \downarrow & & \mathbf{\Phi}^* \downarrow \\ C^1\Lambda^\ell(\widehat{\Omega}) & \xrightarrow{\mathrm{d}_\ell} & C^0\Lambda^{\ell+1}(\widehat{\Omega}) \end{array} .$$

ME in DF :

$$\int_{\partial\Sigma(t)} \mathbf{e}(t) = -\frac{d}{dt}\int_{\Sigma(t)} \mathbf{b}(t) \quad \blacktriangleright \quad \mathrm{d}_1\mathbf{e}(t) = -\frac{d}{dt}\mathbf{b}(t) , \tag{13.1.4.72}$$

$$\int_{\partial\Sigma(t)} \mathbf{h}(t) = \frac{d}{dt}\int_{\Sigma(t)} \mathbf{d}(t) + \int_{\Sigma(t)} \mathbf{j}(t) \quad \blacktriangleright \quad \mathrm{d}_1\mathbf{h}(t) = \frac{d}{dt}\mathbf{d}(t) + \mathbf{j}(t) . \tag{13.1.4.73}$$

in 3D VP :

$$\operatorname{curl} \check{\underline{e}} = -\partial_t \check{\underline{b}}$$
$$\operatorname{curl} \check{\underline{h}} = \partial_t \underline{d} + \vec{j}$$

Beyond Maxwell : heat conduction

$$\int_{\partial V} \check{j} \cdot n \, dS = \int_{\check{V}} f \, dx \Rightarrow d_{d-1} j = f$$

$V \hat{=}$ volume

j: 2-form, f: 3-form

§ 13.1.4.76 Continuity equation

$$d_2 \cdot \Big| \qquad \mathrm{d}_1\mathbf{h}(t) = \frac{d}{dt}\mathbf{d}(t) + \mathbf{j}(t)\,, \tag{13.1.4.73}$$

$$-\frac{d}{dt}\underbrace{\mathrm{d}_2\mathbf{d}(t)}_{\text{charge density 3-form } q} = \mathrm{d}_2\mathbf{j}(t)\,. \tag{13.1.4.79}$$

$$-\frac{d}{dt}\operatorname{div}\check{\mathbf{d}}(t) = \boxed{-\frac{d}{dt}q(t) = \operatorname{div}\check{\mathbf{j}}(t)}\,. \tag{13.1.4.80}$$

13.1.4.3 Potentials

$$\mathcal{R}(d_\ell) \overset{?}{\subseteq} \mathcal{N}(d_{\ell+1})$$

topological assumption

Theorem 13.1.4.81. Existence of potentials

Let $\Omega \subset \mathbb{R}^d$ be a domain and $1 \leq \ell \leq d$. If

$\Big\{$ *every oriented, compact and closed*[a] *ℓ-dimensional sub-manifold $\in \mathcal{S}_\ell(\Omega)$ is the boundary of an $\ell+1$-dimensional sub-manifold $\in \mathcal{S}_{\ell+1}(\Omega)$,*

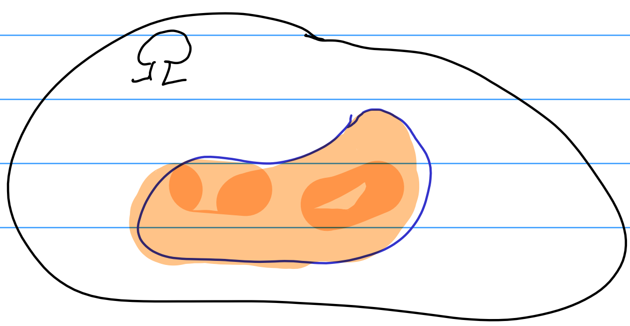
then

$$\omega \in C^1\Lambda^\ell(\Omega)\,, \quad \mathrm{d}_\ell\omega = 0 \quad \Rightarrow \quad \exists \eta \in C^1\Lambda^{\ell-1}(\Omega) \quad \text{such that} \quad \mathrm{d}_{\ell-1}\eta = \omega\,. \tag{13.1.4.82}$$

[a]We call $\Gamma \in \mathcal{S}_\ell(\Omega)$ **closed**, if $\partial\Gamma = \emptyset$

a potential

$\to$ $d=2$, $\ell=1$



3D & VP :

- $\ell = 1$: If

every closed oriented curve in Ω is the boundary of an orientable 2-surface $\subset \Omega$, (T_1)

then

$$\mathbf{u} \in C^1(\Omega, \mathbb{R}^3)\,,\quad \mathbf{curl}\,\mathbf{u} = \mathbf{0} \quad \Rightarrow \quad \exists v \in C^2(\Omega) \text{ such that } \mathbf{grad}\, v = \mathbf{u}\,. \tag{13.1.4.84}$$

Such a v is often called a **scalar potential** of $\mathbf{u}$.

- $\ell = 2$: If

every 2-surface $\subset \Omega$ without boundary is the boundary of some volume $\subset \Omega$, (T_2)

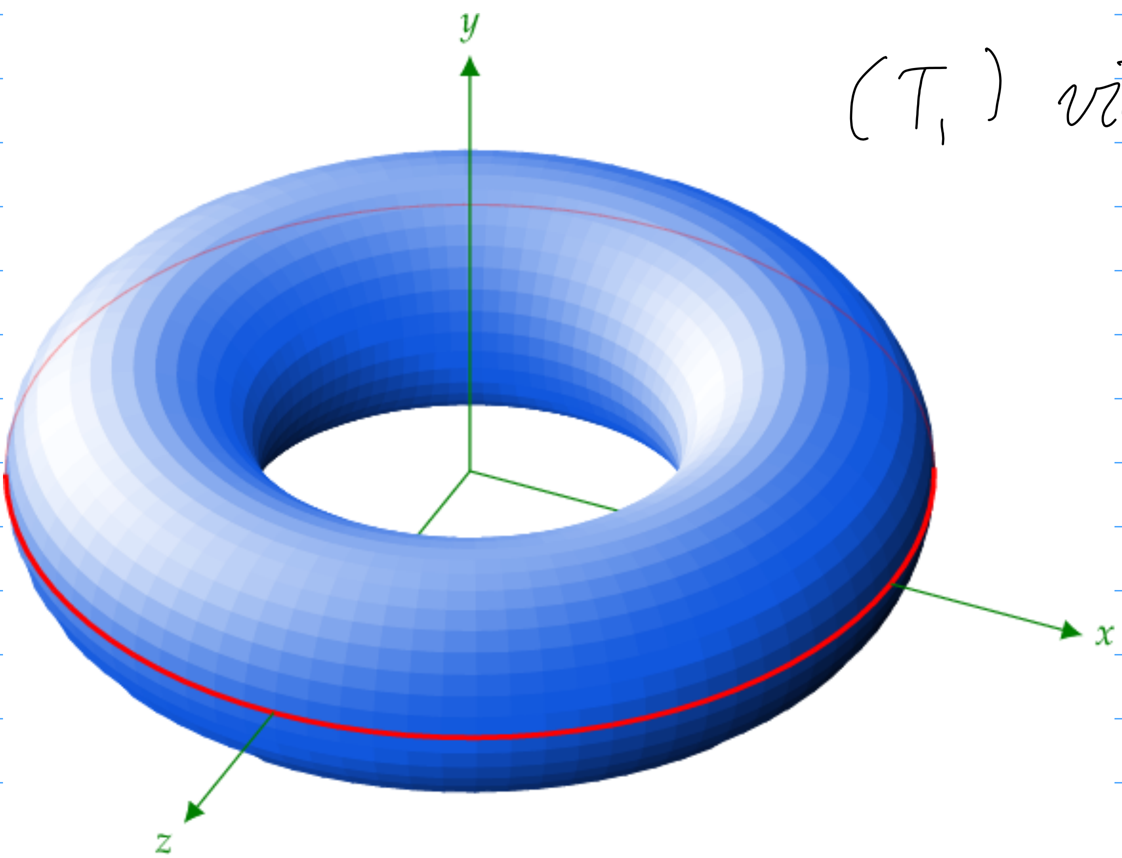
then

$$\mathbf{u} \in C^1(\Omega, \mathbb{R}^3)\,,\quad \operatorname{div}\mathbf{u} = 0 \quad \Rightarrow \quad \exists \mathbf{v} \in C^2(\Omega, \mathbb{R}^3) \text{ such that } \mathbf{curl}\,\mathbf{v} = \mathbf{u}\,. \tag{13.1.4.85}$$

This vector field $\mathbf{v}$ is called a **vector potential** of $\mathbf{u}$.

- $\ell = 3$: For any Ω holds true:

$$\forall u \in C^0(\Omega) \quad \exists \mathbf{v} \in C^1(\Omega, \mathbb{R}^3) \text{ such that } \operatorname{div}\mathbf{v} = u\,. \tag{13.1.4.86}$$



(T_1) violated

EM application: Magnetic vector potential

$d_2 b = 0 \underset{\Omega = \mathbb{R}^3}{\Longrightarrow} \exists$ 1-form a : $b = d_1 a$

not unique!

13.1.4.4 Exterior product

$$u := \frac{1}{2}\left(\check{\mathbf{e}} \cdot \check{\mathbf{d}} + \check{\mathbf{b}} \cdot \check{\mathbf{h}}\right), \qquad (13.1.4.99)$$

$\hookrightarrow$ energy density $\longleftrightarrow$ 3-form

$$\operatorname{div}(\check{\mathbf{e}} \times \check{\mathbf{h}}) + \check{\mathbf{j}} \cdot \check{\mathbf{e}} = -\frac{\partial u}{\partial t}. \qquad (13.1.4.100)$$

[Poynting's theorem]

$\rightarrow$ Need a multiplication operation

Definition 13.1.4.102. Exterior product/wedge product of alternating multilinear forms

For a vector space V the **exterior product** (also called **wedge product**)

$$\bigwedge : \Lambda^\ell(V) \times \Lambda^k(V) \to \Lambda^{\ell+k}(V)$$

($\Lambda^n(V)$ is the vector space of alternating n-linear forms on V, see Def. 13.1.3.4) is defined as

$$(\alpha \wedge \beta)(\mathbf{v}_1, \ldots, \mathbf{v}_{\ell+k}) = \frac{1}{\ell! k!} \sum_{\sigma \in \Pi_{\ell+k}} \operatorname{sgn}(\sigma)\, \alpha(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(\ell)}) \beta(\mathbf{v}_{\sigma(\ell+1)}, \ldots, \mathbf{v}_{\sigma(\ell+k)})$$

for all $\mathbf{v}_1, \ldots, \mathbf{v}_{\ell+k} \in V$.]

Permutations of $(1, \ldots, \ell+k)$

$\wedge$ is associative, bilinear and (anti-)commutative

$$\alpha \wedge \beta = (-1)^{\ell k} (\beta \wedge \alpha)$$

3D V.P.

$$\mathrm{v.p.}(\omega\wedge\eta) = \begin{cases} \mathrm{v.p.}(\omega)\ \mathrm{v.p.}(\eta) & \text{for } \omega\in C^0\Lambda^0(\Omega),\ \eta\in C^0\Lambda^k(\Omega),\ k=0,1,2,3\,, \\ \mathrm{v.p.}(\omega)\times\mathrm{v.p.}(\eta) & \text{for } \omega\in C^0\Lambda^1(\Omega),\ \eta\in C^0\Lambda^1(\Omega)\,, \\ \mathrm{v.p.}(\omega)\cdot\mathrm{v.p.}(\eta) & \text{for } \omega\in C^0\Lambda^1(\Omega),\ \eta\in C^0\Lambda^2(\Omega)\,. \end{cases} \tag{13.1.4.113}$$

How does $\wedge$ interact with d & Φ^*

Theorem 13.1.4.117. Pullback and exterior product/wedge product

Let $\widehat{\Omega},\Omega\subset\mathbb{R}^d$ be two domains connected by a C^1-mapping $\mathbf{\Phi}:\widehat{\Omega}\to\Omega$. Then the pullback $\mathbf{\Phi}^*$ of differential forms commutes with the exterior product of differential forms:

$$\mathbf{\Phi}^*(\omega\wedge\eta) = (\mathbf{\Phi}^*\omega)\wedge(\mathbf{\Phi}^*\eta)\quad \forall\omega\in C^0\Lambda^\ell(\Omega),\ \eta\in C^0\Lambda^k(\Omega),\ \ell,k\in\mathbb{N}_0\,. \tag{13.1.4.118}$$

Theorem 13.1.4.119. Product rule for exterior derivatives

For any domain $\Omega\subset\mathbb{R}^d$ and $\ell,k\in\{0,\dots,d\}$

$$\mathrm{d}_{\ell+k}(\omega\wedge\eta) = \mathrm{d}_\ell\omega\wedge\eta + (-1)^\ell\omega\wedge\mathrm{d}_k\eta\quad \forall\omega\in C^1\Lambda^\ell(\Omega),\ \eta\in C^1\Lambda^k(\Omega)\,. \tag{13.1.4.120}$$

+ gen. Stokes thm.

$$\int_{\partial\Gamma}\omega\wedge\eta = \int_\Gamma \mathrm{d}_\ell\omega\wedge\eta + (-1)^\ell\omega\wedge\mathrm{d}_k\eta\quad \forall\omega\in C^1\Lambda^\ell(\Omega),\ \eta\in C^1\Lambda^k(\Omega)\,, \tag{13.1.4.124}$$

$\Gamma\in\mathcal{S}_{\ell+k+1}(\Omega)$

$\hat{=}$ gen. i.b.p.

For 3D V.P. :

$\ell = 0$, $k = 2$:

$$\boxed{\int_{\partial V} u(\boldsymbol{x})\,\mathbf{v}(\boldsymbol{x})\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}) = \int_V \mathbf{grad}\,u(\boldsymbol{x})\cdot\mathbf{v}(\boldsymbol{x}) + u(\boldsymbol{x})\,\operatorname{div}\mathbf{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}\,. \tag{13.1.4.126}$$

$\hat{=}$ Green's thm.

$\ell = 1$, $k = 1$:

$$\boxed{\int_{\partial V} (\mathbf{u}(\boldsymbol{x})\times\mathbf{v}(\boldsymbol{x}))\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}) = \int_V \mathbf{curl}\,\mathbf{u}(\boldsymbol{x})\cdot\mathbf{v}(\boldsymbol{x}) - \mathbf{u}(\boldsymbol{x})\cdot\mathbf{curl}\,\mathbf{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}\,. \tag{13.1.4.127}$$