ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

# Course Video
# Section 13.2.1: Cochain Calculus

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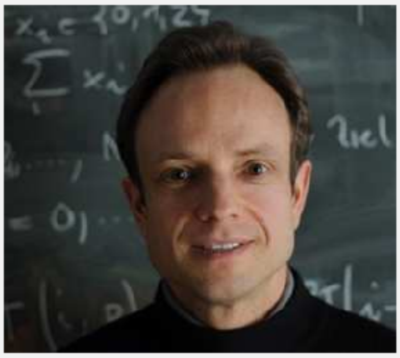
Date: May 3, 2025





**Prerequisites.**

- Elementary linear algebra, matrix calculus

**Dependencies.** [Lecture → Section 2.5.1], [Lecture → Section 2.7.2.2], [Lecture → Section 13.1.2], [Lecture → Section 13.1.4]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

# XIII Finite-Element Exterior Calculus (FEEC)

## 13.2 Discrete Differential Forms

Goal: FE subspaces of $H\Lambda^\ell(d,\Omega)$, $0 \le \ell \le d$

### 13.2.1. Cochain Calculus

$\longrightarrow$ discrete integral forms

#### 13.2.1.1. Meshes and Cochain

IF : $\underline{\omega} : \mathcal{S}_\ell(\Omega) \longrightarrow \mathbb{R}$
(Def 13.1.2.7)

Introduce **discrete integral $\ell$-forms** as mappings from a *finite* subset of $\mathcal{S}_\ell(\Omega)$ into $\mathbb{R}$.

suitable : meaningful $\partial$

$\triangleright$ Use FE mesh (Def 2.5.1.1)

**Definition 13.2.1.1. Generalized oriented meshes**

A **generalized oriented mesh** $\mathcal{M}$ of a domain $\Omega \subset \mathbb{R}^d$ is a collection of sets of **$\ell$-facets**,

$$\mathcal{M} := (\mathcal{F}_\ell(\mathcal{M}))_{\ell=0}^{d}$$

such that

(i) $\mathcal{F}_\ell(\mathcal{M}) \subset \mathcal{S}_\ell(\Omega)$: every $\ell$-facet $f \in \mathcal{F}_\ell(\mathcal{M})$ is an *oriented* (*), relatively *open*, bounded, $\ell$-dimensional, *non-degenerate*[a] $C^1_{\mathrm{pw}}$-surface $\subset \mathbb{R}^d$,

(ii) the facets form a *partition* of $\overline{\Omega}$:

$$\overline{\Omega} = \bigcup_{\ell=0}^{d} \bigcup_{F \in \mathcal{F}_\ell(\mathcal{M})} F \,, \quad f \cap f' = \emptyset \quad \forall f \in \mathcal{F}_j(\mathcal{M}),\ f' \in \mathcal{F}_k(\mathcal{M}),\ k,j \in \{0,\dots,d\}\,,$$

(iii) for every $F \in \mathcal{F}_\ell(\mathcal{M}), 0 < \ell \leq d$, its boundary $\partial F$ is the union of the closures of finitely many $\ell - 1$-facets:

$$\forall F \in \mathcal{F}_\ell(\mathcal{M}): \quad \exists\{f_1,\dots,f_m\} \subset \mathcal{F}_{\ell-1}(\mathcal{M}) \quad \text{such that} \quad \partial F = \overline{f}_1 \cup \dots \cup \overline{f}_m\,,$$

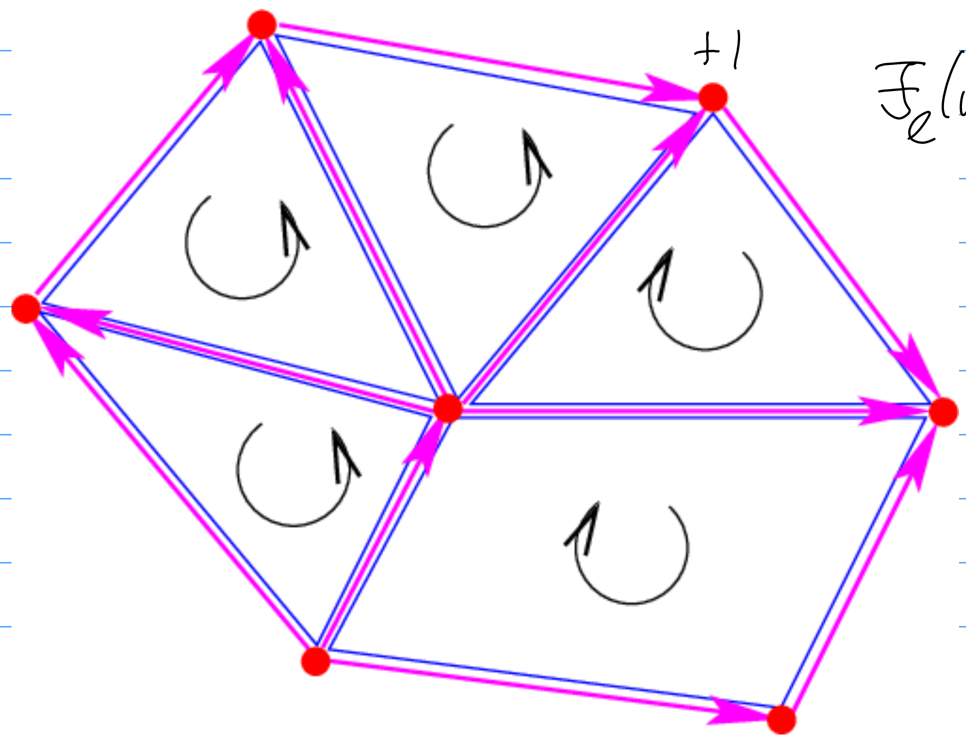
(iv) the intersection of the closures of any two $\ell$-facets is an $\ell - 1$-facet,

$$\forall \ell \in \{1,\dots,d\}\,, \quad \forall f, f' \in \mathcal{F}_\ell(\mathcal{M}): \quad \overline{f} \cap \overline{f}' \in \mathcal{F}_{\ell-1}(\mathcal{M})\,,$$

(v) for every $f \in \mathcal{F}_\ell(\mathcal{M}), 0 \leq \ell < d$, there is a $F \in \mathcal{F}_{\ell+1}(\mathcal{M})$ such that $f \subset \partial F$.

[a] An $\ell$-dimensional sub-manifold $\subset \mathbb{R}^d$ is called non-degenerate, if it has non-zero $\ell$-dimensional volume.

Ex: $d = 2$




$$\mathcal{F}_\ell(\mathcal{M}) = \begin{cases} \text{cells}, & \ell = 2 \\ \text{edge}, & \ell = 1 \\ \text{node}, & \ell = 0 \end{cases}$$

Special case : simplicial meshes*

$d=2$ : triangular , $d=3$ tetrahedral

* All $\ell$-facets are $\ell$-simplices = convex hull of $\ell+1$ nodes.

$\Sigma\,[x_0, \dots, x_{\ell+1}]$

$\uparrow$ ordering determines orientation

**Definition 13.2.1.7. Cochains**

Given a generalized oriented mesh $\mathcal{M} = (\mathcal{F}_\ell)_{\ell=0}^d$ of a domain $\Omega \subset \mathbb{R}^d$ an **$\ell$-cochain** $\vec{\omega}$ is a mapping $\vec{\omega} : \mathcal{F}_\ell(\mathcal{M}) \to \mathbb{R}$.

Notation : $C^\ell(\mathcal{M}) \mathrel{\hat{=}}$ vector space

$\langle \vec{\omega}, f \rangle := \vec{\omega}(f)$ , $\vec{\omega} \in C^\ell(\mathcal{M})$, $f \in \mathcal{F}_\ell(\mathcal{M})$

$\mathcal{F}_\ell(\mathcal{M}) = \{f_1, \dots, f_{N_\ell}\}$, $N_\ell = \# \mathcal{F}_\ell(\mathcal{M})$

$\rightarrow$ $C^\ell(\mathcal{M}) \cong \mathbb{R}^{N_\ell}$ by $(\vec{\omega})_i := \langle \vec{\omega}, f_i \rangle$

## 13.2.1.2. Discrete exterior Derivative

IF : $\langle \underline{d_\ell \omega}, T \rangle = \langle \underline{\omega}, \partial T \rangle$ , $T \in S_{\ell+1}(\Omega)$

$\uparrow$ w/ induced orientation

CC: $\vec{\omega} \in C^\ell(\mathcal{M})$

additive

$$\langle d_\ell \vec{\omega}, F \rangle = \langle \vec{\omega}, \partial F \rangle = \sum_{i=1}^{m} \pm \langle \vec{\omega}, f_i \rangle$$

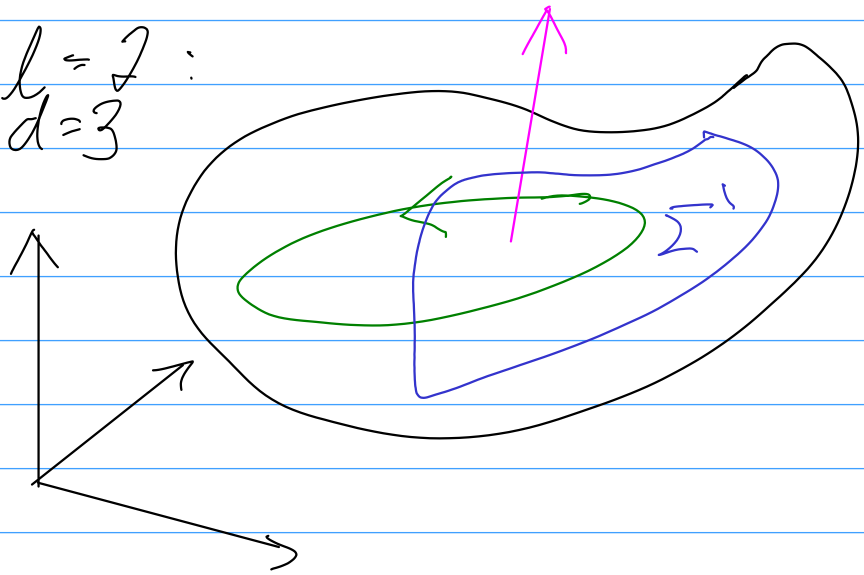
$\forall F \in \mathcal{F}_{\ell+1}(\mathcal{M})$

Def. $\mathcal{M} \Rightarrow \partial F = \bar{f}_1 \cup \dots \cup \bar{f}_m$, $f_i \in \mathcal{F}_\ell(\mathcal{M})$

§ 13.2.1.8 Induced orientation of mesh facets

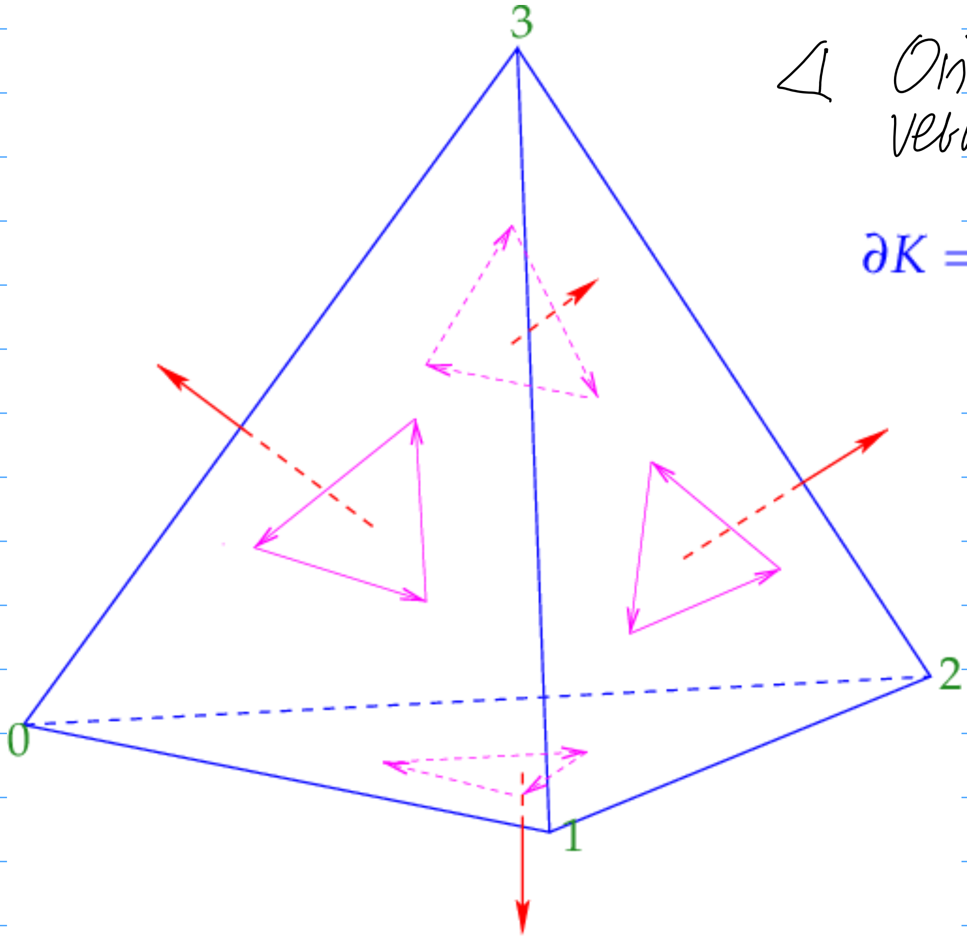
$\Sigma, \Sigma' \in \mathcal{S}_\ell(\Omega)$ : $\Sigma' \subset \Sigma \Rightarrow \Sigma'$ inherits orientation

$\ell = 2$ :
$d = 3$




$F \in \mathcal{F}_{\ell+1}(\mathcal{M}) \subset \mathcal{S}_{\ell+1}(\mathcal{M}) \Rightarrow f_i \subset \partial F$ inherits orientation from $\partial F$
[induced orientation]

Ex: Simplex in 3D




◁ Oriented by numbering vertices

$$\partial K = \Sigma[1,2,3] \cup \Sigma[0,3,2] \cup \Sigma[0,1,3] \cup \Sigma[0,2,1]\,,$$

↑ w/ correct induced orientation

General combinatorial formula for simplex

$$\partial\Sigma[\boldsymbol{x}_0,\dots,\boldsymbol{x}_\ell] = \bigcup_{k=0}^{\ell} \underbrace{(-1)^k\,\Sigma[\boldsymbol{x}_0,\dots,\cancel{\boldsymbol{x}_k},\dots,\boldsymbol{x}_\ell]}_{\text{face-simplex w/ induced orientation}}\,, \tag{13.2.1.11}$$

§ 13.2.1.10 Relative orientation

↓

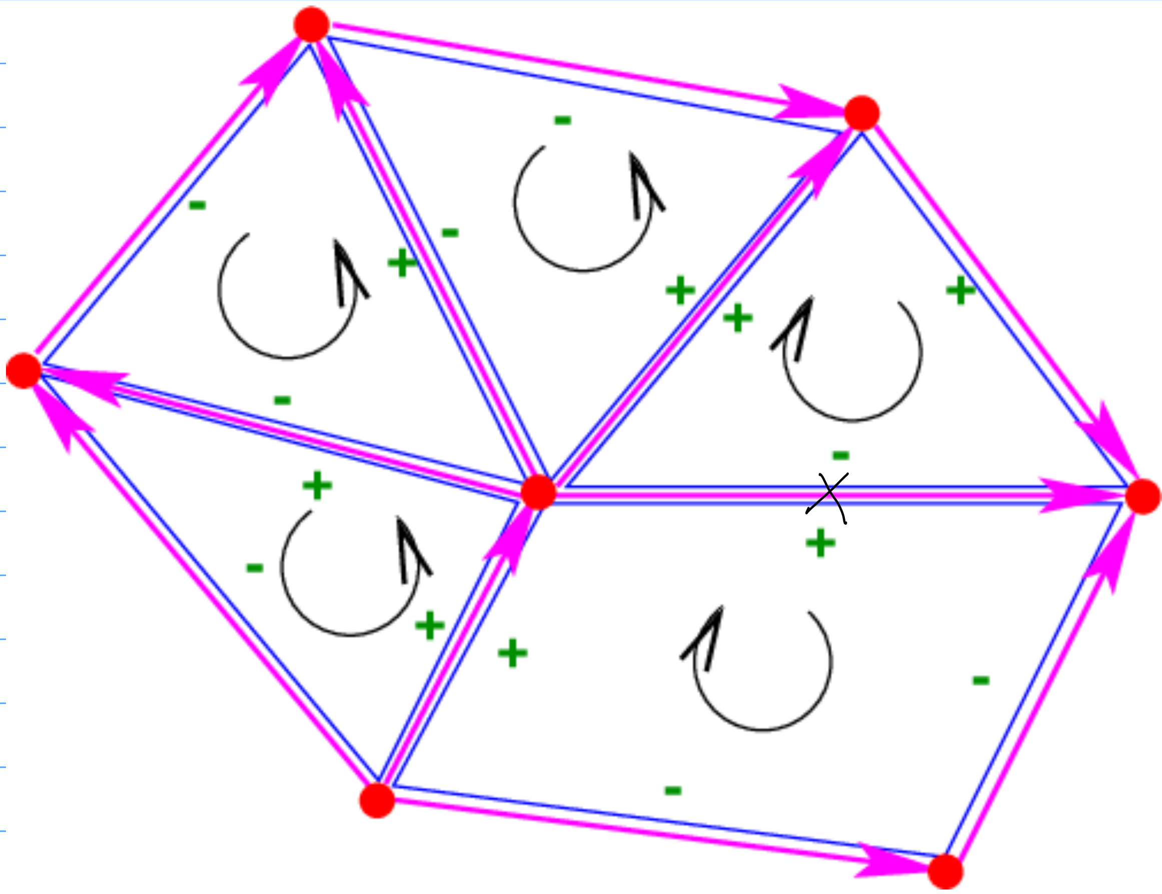
mismatch : intrinsic or. ⟺ induced or.

**Definition 13.2.1.13. Relative orientation of mesh facets**

Given $F \in \mathcal{F}_\ell(\mathcal{M})$, $0 < \ell \leq d$, and $f \in \mathcal{F}_{\ell-1}(\mathcal{M})$, we define their **relative orientation** as

$$o_r(f,F) := \begin{cases} 1 & \text{, if } f \subset F \text{ and the orientations of } f \text{ and } \partial F \text{ agree,} \\ -1 & \text{, if } f \subset F \text{ and the orientations of } f \text{ and } \partial F \text{ are opposite,} \\ 0 & \text{, if } f \not\subset \partial F. \end{cases}$$

2D example:




**Definition 13.2.1.16. Exterior derivative for cochains**

Given a generalized oriented mesh $\mathcal{M}$ of $\Omega \subset \mathbb{R}^d$ and $\ell \in \{0,\dots,d-1\}$, the **exterior derivative** $\vec{\mathsf{d}}_\ell : \mathcal{C}^\ell(\mathcal{M}) \to \mathcal{C}^{\ell+1}(\mathcal{M})$ is defined by

$$\left\langle \vec{\mathsf{d}}_\ell \vec{\omega}, F \right\rangle = \sum_{f \in \mathcal{F}_\ell(\mathcal{M})} o_r(f,F) \langle \vec{\omega}, f \rangle \quad \forall F \in \mathcal{F}_{\ell+1}(\mathcal{M}) \,. \tag{13.2.1.17}$$

→ linear operator

$$C^\ell(\mathcal{M}) \cong \mathbb{R}^{N_\ell}, \quad N_\ell = \#\mathcal{F}_\ell(\mathcal{M})$$

Matrix representation $\mathbf{D}_\ell$ of $\vec{\mathrm{d}}_\ell$:

$$\left(\vec{\mathrm{d}}_\ell\vec{\omega} = \mathbf{D}_\ell\vec{\omega}\right)_i = \left\langle \vec{\mathrm{d}}_\ell\vec{\omega}, F_i \right\rangle \overset{\text{def}}{=} \sum_{f\in\mathcal{F}_\ell(\mathcal{M})} o_r(f,F_i)\langle\vec{\omega},f\rangle = \sum_{j=1}^{N_\ell} o_r(f_j,F_i)(\vec{\omega})_j \,.$$

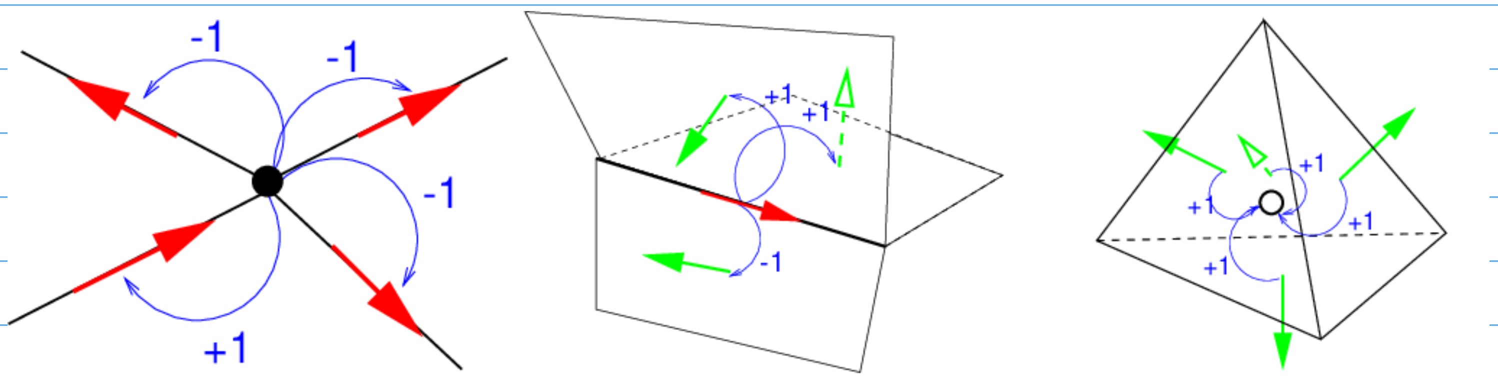
M x V-prod.!

$$\mathbf{D}_\ell = \left[o_r(f_j,F_i)\right]_{\substack{i=1,\dots,N_{\ell+1}\\ j=1,\dots,N_\ell}} \in \{-1,0,1\}^{N_{\ell+1},N_\ell} \qquad \begin{aligned}\mathcal{F}_\ell(\mathcal{M}) &= \{f_1,\dots,f_{N_\ell}\}\,,\\ \mathcal{F}_{\ell+1}(\mathcal{M}) &= \{F_1,\dots,F_{N_{\ell+1}}\}\,.\end{aligned} \tag{13.2.1.18}$$

$\Rightarrow$ $\mathbf{D}_\ell$ sparse

$\mathbf{D}_\ell$ : (signed) incidence matrices of $\mathcal{M}$

Stencil view:

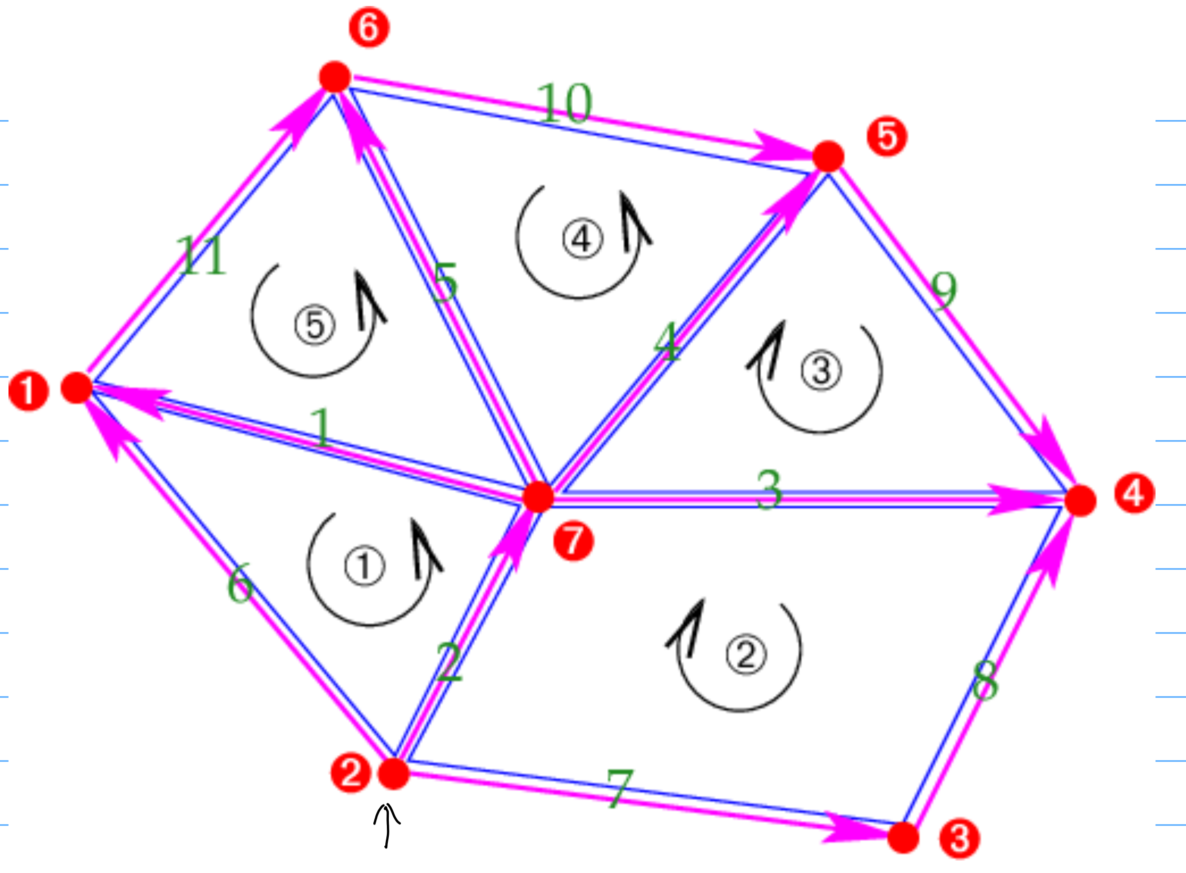



$\mathbf{D}_0$: node-edge incidence matrix — [discrete grad]

$\mathbf{D}_1$: edge-face incidence matrix — [discrete curl]

$\mathbf{D}_2$: face-cell incidence matrix — [discrete div]

Ex: $d = 2$



“node-edge”:

$$\mathbf{D}_0 = \underbrace{\left.\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\right\}}_{\sim \text{nodes}} \sim \text{edges}$$

“edge-cell”:

$$\mathbf{D}_1 = \underbrace{\left.\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}\right\}}_{\sim \text{edges}} \sim \text{cells}$$

# Properties of $\vec{d}_\ell$

**Theorem 13.2.1.21.** $\vec{\mathrm{d}} \circ \vec{\mathrm{d}} = 0$

*For any generalized mesh* $\mathcal{M}$ *of a domain* $\Omega \subset \mathbb{R}^d$ *and* $\ell \in \{0, \dots, d-2\}$ *holds true*

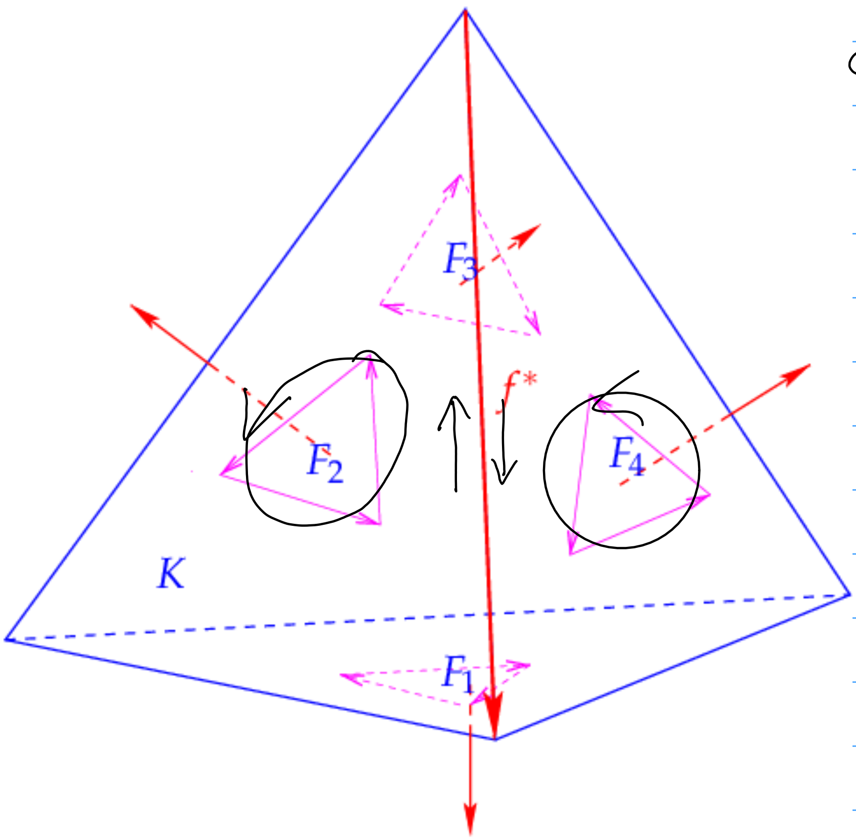
$$\vec{\mathrm{d}}_{\ell+1} \circ \vec{\mathrm{d}}_\ell = 0 \quad \Leftrightarrow \quad \mathbf{D}_{\ell+1}\mathbf{D}_\ell = \mathbf{O} \quad \Leftrightarrow \quad \mathcal{R}(\mathbf{D}_\ell) \subset \mathcal{N}(\mathbf{D}_{\ell+1}) \, .$$

Idea of proof $(d=3, \ell=1)$

$f^* \in \mathcal{F}_1(\mathcal{M})$, $K \in \mathcal{F}_3(\mathcal{M})$, $f^* \subset \partial K$

Compute $(\mathbf{D}_2 \mathbf{D}_1)_{K,f^*} = 0$

cancellation of rel. ori. !




$\ell = 1$

**Theorem 13.1.4.81. Existence of potentials**

*Let $\Omega \subset \mathbb{R}^d$ be a domain and $1 \leq \ell \leq d$. If*

*every oriented, compact and closed $\ell$-dimensional sub-manifold of $\in \mathcal{S}_\ell(\Omega)$ is the boundary of an $\ell+1$-dimensional sub-manifold $\in \mathcal{S}_{\ell+1}(\Omega)$,*

*then*

$$\omega \in C^1\Lambda^\ell(\Omega)\,,\quad \mathrm{d}_\ell\omega = 0 \quad \Rightarrow \quad \exists \eta \in C^1\Lambda^{\ell-1}(\Omega) \quad \text{such that} \quad \mathrm{d}_{\ell-1}\eta = \omega\,. \tag{13.1.4.82}$$

topological assumption

**Theorem 13.2.1.22. Existence of cochain potentials**

*Let the domain $\Omega \subset \mathbb{R}^d$ satisfy the assumptions of Thm. 13.1.4.81. Then for any generalized oriented mesh $\mathcal{M}$ of $\Omega$ and for any $1 \leq \ell \leq d$ we have*

$$\begin{aligned} &\vec{\omega} \in \mathcal{C}^\ell(\mathcal{M}):\ \vec{\mathrm{d}}_\ell\vec{\omega} = 0 \quad \Rightarrow \quad \exists \vec{\eta} \in \mathcal{C}^{\ell-1}(\mathcal{M}) \quad \text{such that} \quad \vec{\mathrm{d}}_{\ell-1}\vec{\eta} = \vec{\omega} \\ &\Leftrightarrow \quad \mathcal{R}(\vec{\mathrm{d}}_{\ell-1}) = \mathcal{N}(\vec{\mathrm{d}}_\ell) \quad \Leftrightarrow \quad \mathcal{R}(\mathbf{D}_{\ell-1}) = \mathcal{N}(\mathbf{D}_\ell)\,. \end{aligned} \tag{13.2.1.23}$$

## 13.2.1.3. Semi-discrete Maxwell's Equations

ME in IF:

$$\underline{\mathrm{d}}_1\underline{\mathbf{e}}(t) = -\frac{d}{dt}\underline{\mathbf{b}}(t) \quad , \quad \underline{\mathrm{d}}_1\underline{\mathbf{h}}(t) = \frac{d}{dt}\underline{\mathbf{d}}(t) + \underline{\mathbf{j}}(t)\,, \tag{13.1.4.30}$$

discrete ME

$$\vec{\mathrm{d}}_1\vec{\mathbf{e}}(t) = -\frac{d}{dt}\vec{\mathbf{b}}(t) \quad , \quad \vec{\mathrm{d}}_1\vec{\mathbf{h}}(t) = \frac{d}{dt}\vec{\mathbf{d}}(t) + \vec{\mathbf{j}}(t)\,, \tag{13.2.1.29}$$

$$\mathbf{D}_1\vec{\mathbf{e}}(t) = -\frac{d}{dt}\vec{\mathbf{b}}(t) \quad , \quad \mathbf{D}_1\vec{\mathbf{h}}(t) = \frac{d}{dt}\vec{\mathbf{d}}(t) + \vec{\mathbf{j}}(t)\,. \tag{13.2.1.30}$$

FL AL source current = given data

A remarkable property:

IF $\longrightarrow$ CC : sampling/measurement

$$\mathsf{S}_\ell : \left\{ \begin{array}{ccc} \mathcal{I}^\ell(\Omega) & \to & \mathcal{C}^\ell(\mathcal{M}) \\ \underline{\omega} & \mapsto & \left(\langle \underline{\omega}, f\rangle\right)_{f\in\mathcal{F}_\ell(\mathcal{M})} \end{array} \right. , \quad \ell \in \{0,1,2,3\}\,. \tag{13.2.1.32}$$

$\Rightarrow$ Sampled Fields

$$\begin{array}{llll} \vec{\mathbf{e}} := \mathsf{S}_1\underline{\mathbf{e}}\,, & \vec{\mathbf{h}} := \mathsf{S}_1\underline{\mathbf{h}}\,, & & \vec{\mathbf{e}},\vec{\mathbf{h}} \in \mathbb{R}^{N_1}\,, \quad N_1 := \sharp\mathcal{F}_1(\mathcal{M})\,, \\ \vec{\mathbf{b}} := \mathsf{S}_2\underline{\mathbf{b}}\,, & \vec{\mathbf{d}} := \mathsf{S}_2\underline{\mathbf{d}}\,, & \vec{\mathbf{j}} := \mathsf{S}_2\underline{\mathbf{j}}\,, & \vec{\mathbf{b}},\vec{\mathbf{d}},\vec{\mathbf{j}} \in \mathbb{R}^{N_2}\,, \quad N_2 := \sharp\mathcal{F}_2(\mathcal{M})\,. \end{array}$$

These vectors will be exact solution of the algebraic Maxwell's equations ~~??~~! (30)

Well in (30):

$\#\text{unknowns} = 2 \cdot \# \mathcal{F}_1(\mathcal{M}) + 2 \cdot \# \mathcal{F}_2(\mathcal{M})$

$\#\text{equs.} = 2 \cdot \# \mathcal{F}_2$

(30) is strongly under-determined