ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 13.2.2: Whitney Forms

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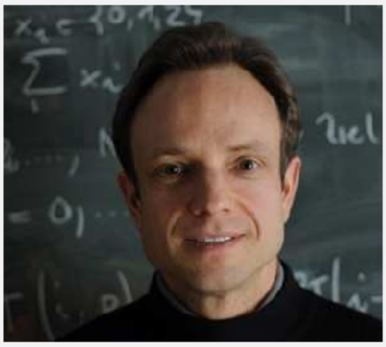
(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Elementary calculus

Dependencies. [Lecture → Section 2.6.1] plus all of [Lecture → Section 13.1] and [Lecture → Section 13.2.1]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XIII Finite-Element Exterior Calculus (FEEC)

13.2 Discrete Differential Forms

13.2.2. Whitney Forms

13.2.2.1 Generalized Interpolation

Goal:

ℓ-cochain $\vec{\omega}$ (mapping $\mathcal{F}_\ell(\mathcal{M}) \to \mathbb{R}$) $\longrightarrow$ *matching* integral ℓ-form $\underline{\omega}$ (mapping $\mathcal{S}_\ell(\Omega) \to \mathbb{R}$)

Interpolation condition $\langle \underline{\omega}, f \rangle = \langle \vec{\omega}, f \rangle \quad \forall f \in \mathcal{F}_\ell(\mathcal{M})$

$\mathcal{M} \hat{=}$ simplicial mesh of $\Omega \subset \mathbb{R}^d$

$\ell = 0$: Seek C^0-function interpolating values in the nodes of the mesh

(local) p.w.-linear interpolation $C^0(\Omega) \to \mathcal{S}_1^0(\mathcal{M})$

$$\langle \underline{\omega}_{|K}, x \rangle = \sum \lambda_i(x) \langle \vec{\omega}, a_j \rangle \qquad (*)$$

[λ_i = barycentric coords.] on $K = \Sigma[a_j]_{j=0}^{d+1}$

$\to$ Locality: $\forall K \in \mathcal{F}_d(\mathcal{M})$ demand

$$\langle \vec{\omega}, f \rangle = 0 \quad \forall f \in \mathcal{F}_\ell(\mathcal{M}),\ f \subset \partial K \quad \Rightarrow \quad \underbrace{\mathsf{W}_\ell \vec{\omega}}_{= \underline{\omega}}|_K = 0 : \qquad (13.2.2.8)$$

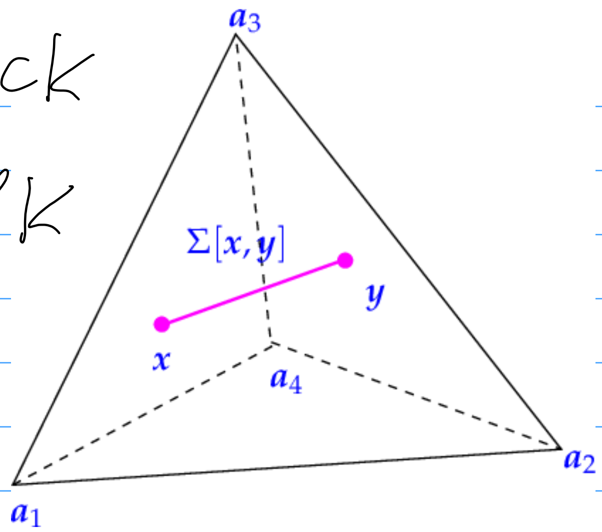
13.2.2.2 Local construction

$\ell = 0$

$$\langle \underline{\omega}_K, \underline{x} \rangle = \sum_{j=0}^{d} \lambda_i(x) \langle \vec{\omega}, \underline{a}_j \rangle, \quad \underline{x} \in K$$

$\to$ x as a linear combination of the vertices of K

$\ell = 1$: line segment $[x,y] \subset K$ as a "linear combination" of edges of K



$$\begin{aligned}\Sigma[\boldsymbol{x},\boldsymbol{y}] &= \{t\boldsymbol{x} + (1-t)\boldsymbol{y}\,;\, 0 \le t \le 1\} \\ &= \Big\{\sum_i (t\lambda_i(\boldsymbol{x}) + (1-t)\lambda_i(\boldsymbol{y}))\boldsymbol{a}_i\,;\, 0\le t\le 1\Big\} \\ &= \Big\{\sum_i \Big(t\sum_j \lambda_j(\boldsymbol{y})\lambda_i(\boldsymbol{x}) + (1-t)\sum_j \lambda_j(\boldsymbol{x})\lambda_i(\boldsymbol{y})\Big)\boldsymbol{a}_i\,;\, 0\le t\le 1\Big\} \\ &= \Big\{\sum_i\sum_j \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})(t\boldsymbol{a}_i + (1-t)\boldsymbol{a}_j)\,;\, 0\le t\le 1\Big\}.\end{aligned} \tag{13.2.2.12}$$

[use barycentric rep. $x = \sum_{i=0}^{d+1} \lambda_i(x) a_i$]

$\sum \lambda_i \equiv 1$

$$\begin{aligned}\langle \underline{\mathsf{W}}_1|_K\vec{\omega}, \Sigma[\boldsymbol{x},\boldsymbol{y}]\rangle &:= \sum_{i=1}^{d+1}\sum_{j=1}^{d+1} \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})\langle\vec{\omega}, \Sigma[\boldsymbol{a}_i,\boldsymbol{a}_j]\rangle \\ &= \sum_{i<j} (\lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y}) - \lambda_i(\mathbf{y})\lambda_j(\mathbf{x}))\langle\vec{\omega}, \Sigma[\boldsymbol{a}_i,\boldsymbol{a}_j]\rangle\,, \quad \boldsymbol{x},\boldsymbol{y}\in K\,.\end{aligned} \tag{13.2.2.13}$$

local interpolation operator $C^1(K) \to \mathcal{I}^1(K)$

Next step: localization IF $\longrightarrow$ DF

$$\omega(\boldsymbol{x})(\mathbf{v}_1,\dots,\mathbf{v}_\ell) := \lim_{t\to 0}\frac{\ell!}{t^\ell}\langle\omega, S_t(\boldsymbol{x})\rangle\,,\quad S_t(\boldsymbol{x}) := \Sigma[\boldsymbol{x}, \boldsymbol{x}+t\mathbf{v}_1,\dots,\boldsymbol{x}+t\mathbf{v}_\ell]\,, \tag{13.2.2.16}$$

$\ell = 1$:
$$\begin{aligned}(\mathsf{W}_{1,K}\vec{\omega})(\boldsymbol{x})(\boldsymbol{v}) &= \lim_{t\to 0}\frac{1}{t}\langle\underline{\mathsf{W}}_{1,K}\vec{\omega}, \Sigma[\boldsymbol{x},\boldsymbol{x}+t\boldsymbol{v}]\rangle \\ &= \lim_{t\to 0}\sum_{i<j}\Big(\lambda_i(\boldsymbol{x})\frac{\lambda_j(\boldsymbol{x}+t\boldsymbol{v}) - \lambda_j(\boldsymbol{x})}{t} - \lambda_j(\boldsymbol{x})\frac{\lambda_i(\boldsymbol{x}+t\boldsymbol{v}) - \lambda_i(\boldsymbol{x})}{t}\Big)\langle\vec{\omega}, \Sigma[\boldsymbol{a}_i,\boldsymbol{a}_j]\rangle \\ &= \sum_{i<j}\underbrace{(\lambda_i(\boldsymbol{x})\mathrm{d}_0\lambda_j(\boldsymbol{x})(\boldsymbol{v}) - \lambda_j(\boldsymbol{x})\mathrm{d}_0\lambda_i(\boldsymbol{x})(\boldsymbol{v}))}_{\text{basis 1-form = local Whitney 1-form}}\langle\vec{\omega}, \Sigma[\boldsymbol{a}_i,\boldsymbol{a}_j]\rangle\,.\end{aligned}$$

$\in C^\infty\Lambda^1(K)$

General formula for d-simplex K:

$$\boxed{(\mathsf{W}_{\ell,K}\vec{\omega}) = \ell! \sum_{f\in\mathcal{F}_\ell(K)} \underbrace{\sum_{j=0}^{\ell}(-1)^j \left(\lambda_{i_j}\, \mathrm{d}_0\lambda_{i_0}\wedge\cdots\wedge \cancel{\mathrm{d}_0\lambda_{i_j}}\wedge\cdots\wedge \mathrm{d}_0\lambda_{i_\ell}\right)}_{\beta^\ell_{f,K}} \cdot \langle\vec{\omega}, f\rangle}\ , \qquad (13.2.2.17)$$

ℓ-facets of K $\qquad\qquad$ $\to$ Local Whitney ℓ-forms

LIC: $\displaystyle\int_f \mathsf{W}_{\ell,K}\vec{\omega} = \langle\vec{\omega}, f\rangle \quad \forall f\in\mathcal{F}_\ell(K)$

$\mathsf{W}_{\ell,K} \longleftrightarrow \vec{\mathrm{d}}_\ell,\ \mathrm{d}_\ell$?

$\ell = 0$:

$$\begin{aligned}
\int_{\Sigma[\boldsymbol{x},\boldsymbol{y}]} \mathsf{W}_{1,K}(\vec{\mathrm{d}}_0\vec{\omega}) &= \sum_{i=1}^{4}\sum_{j=1}^{4}\lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})(\vec{\mathrm{d}}_0\vec{\omega})(\Sigma[\boldsymbol{a}_i,\boldsymbol{a}_j]) \qquad \downarrow \text{ def. of } \vec{\mathrm{d}}_0\\
&= \sum_{i=1}^{4}\sum_{j=1}^{4}\lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})(\vec{\omega}(\boldsymbol{a}_j)-\vec{\omega}(\boldsymbol{a}_i))\\
&= \sum_{j=1}^{4}\lambda_j(\boldsymbol{y})\vec{\omega}(\boldsymbol{a}_j) - \sum_{i=1}^{4}\lambda_i(\boldsymbol{x})\vec{\omega}(\boldsymbol{a}_i)\\
&= (\mathsf{W}_{0,K}(\vec{\omega}))(\boldsymbol{y}) - (\mathsf{W}_{0,K}(\vec{\omega}))(\boldsymbol{x}) = \int_{\partial\Sigma[\boldsymbol{x},\boldsymbol{y}]}\mathsf{W}_{0,K}\vec{\omega} = \int_{\Sigma[\boldsymbol{x},\boldsymbol{y}]}\mathrm{d}_0\mathsf{W}_{0,K}\vec{\omega}\ ,
\end{aligned}$$

Lemma 13.2.2.22. Local interpolation commutes with exterior derivative

The local interpolant of the exterior derivative of an ℓ-cochain is the same as the exterior derivative of the interpolant of that cochain,

$$\mathrm{d}_\ell(\mathsf{W}_{\ell,K}\vec{\omega}) = \mathsf{W}_{\ell+1,K}(\vec{\mathrm{d}}\vec{\omega}) \quad \begin{array}{l}\forall\vec{\omega}\in\mathcal{C}^\ell(\mathcal{M})\,,\\ \forall K\in\mathcal{F}_d(\mathcal{M})\end{array} \quad \Leftrightarrow \quad \begin{array}{ccc}\mathcal{C}^\ell(\mathcal{M}) & \xrightarrow{\mathsf{W}_{\ell,K}} & C^\infty\Lambda^\ell(K)\\ \vec{\mathrm{d}}_\ell\big\downarrow & & \mathrm{d}_\ell\big\downarrow\\ \mathcal{C}^{\ell+1}(\mathcal{M}) & \xrightarrow{\mathsf{W}_{\ell+1,K}} & C^\infty\Lambda^{\ell+1}(K)\,.\end{array}$$

VP for local WF

$$\boxed{\beta^{\ell}_{f,K} := \ell! \sum_{j=0}^{\ell} (-1)^j \lambda_{i_j}\, \mathrm{d}_0\lambda_{i_0} \wedge \cdots \wedge \cancel{\mathrm{d}_0\lambda_{i_j}} \wedge \cdots \wedge \mathrm{d}_0\lambda_{i_\ell}}$$ for some $f \in \mathcal{F}_\ell(K)$ (13.2.2.20)

+ translation table : $d_0 \longrightarrow$ grad
$\wedge \longrightarrow \cdot$ or $\times$

$\ell = 2$, 2D, $\omega(x)\underline{v} = \underline{u}(x)\cdot\underline{v}$, $\underline{u} := \mathrm{v.p.}(\omega)$

$$\mathrm{v.p.}(\beta^1_{\Sigma[\boldsymbol{a}_i,\boldsymbol{a}_j]}) = \mathrm{v.p.}(\lambda_i \wedge \mathrm{d}_0\lambda_j - \lambda_j \wedge \mathrm{d}_0\lambda_i) = \lambda_i \,\mathbf{grad}\,\lambda_j - \lambda_j\,\mathbf{grad}\,\lambda_i \quad \text{on} \quad K\,. \tag{13.2.2.24}$$

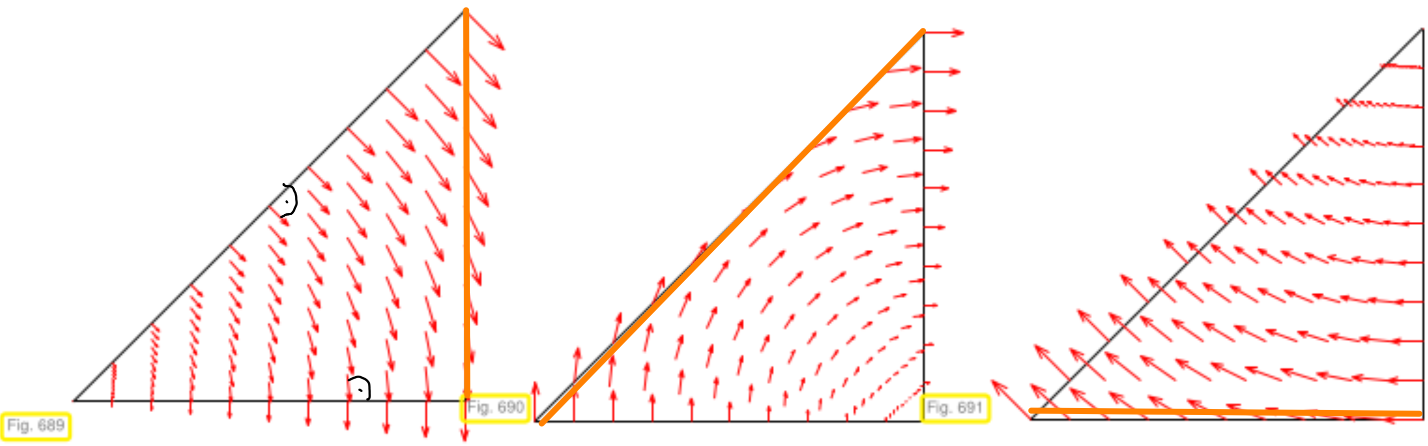


Fig. 689 Fig. 690 Fig. 691

$$\int_\gamma \omega = \int_\gamma \underline{u}\cdot d\vec{s}$$

3D: $\ell = 2$

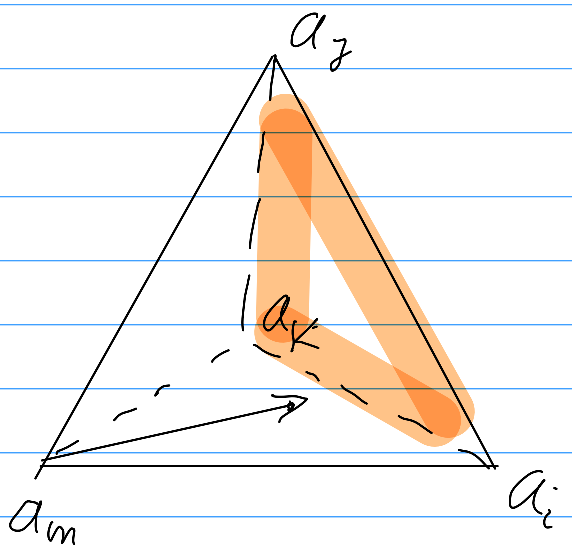
$$\beta^2_{f,K} = 2\big(\lambda_i(\mathrm{d}_0\lambda_j \wedge \mathrm{d}_0\lambda_k) - \lambda_j(\mathrm{d}_0\lambda_i \wedge \mathrm{d}_k\lambda_k) + \lambda_k(\mathrm{d}_0\lambda_i \wedge \mathrm{d}_0\lambda_j)\big)\,, \tag{13.2.2.33}$$

$$f := \Sigma[\boldsymbol{a}_i, \boldsymbol{a}_j, \boldsymbol{a}_k]\,, \quad \{i,j,k\} \subset \{1,2,3,4\}\,. \tag{13.2.2.34}$$

$$\mathrm{v.p.}(\beta^2_{f,K}) = \lambda_i\,\mathbf{grad}\,\lambda_j \times \mathbf{grad}\,\lambda_k + \lambda_j\,\mathbf{grad}\,\lambda_k \times \mathbf{grad}\,\lambda_i + \lambda_k\,\mathbf{grad}\,\lambda_i \times \mathbf{grad}\,\lambda_j\,. \tag{13.2.2.35}$$

Rewrite using formulas for λ_i:

$$\mathrm{v.p.}(\beta^2_{f,K})(\boldsymbol{x}) = \frac{1}{3}\frac{1}{|K|}(\boldsymbol{x}-\boldsymbol{a}_m)\,, \qquad \boldsymbol{x}\in K\,,\quad \{m\}\cup\{i,j,k\}=\{1,2,3,4\}\,. \tag{13.2.2.36}$$



$$\int_F \omega = \int_F \mathrm{v.p.}(\omega)\cdot \boldsymbol{n}\, dS$$

$$\mathrm{span}\{\mathrm{v.p.}(\beta^2_{f,K})\}_{f\in\mathcal{F}_2(K)} = \left\{\boldsymbol{x}\in K\mapsto a\,\boldsymbol{x}+\boldsymbol{b},\ a\in\mathbb{R},\ \boldsymbol{b}\in\mathbb{R}^3\right\}\subset(\mathcal{P}_1(K))^3\,. \tag{13.2.2.37}$$

Degree ℓ	$\mathrm{span}\{\mathrm{v.p.}(\beta^\ell_{f,K})\}_{f\in\mathcal{F}_\ell(K)}$	Cochain coeff.
$\ell=0$	$\{\boldsymbol{x}\in K\mapsto a+\boldsymbol{b}\cdot\boldsymbol{x},\ a\in\mathbb{R},\ \boldsymbol{b}\in\mathbb{R}^3\}$	$u\mapsto u(\boldsymbol{a}_i)$
$\ell=1$	$\{\boldsymbol{x}\in K\mapsto \boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x},\ \boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3\}$	$\mathbf{u}\mapsto\int_{\Sigma[\boldsymbol{a}_i,\boldsymbol{a}_j]}\mathbf{u}\cdot d\vec{\mathbf{s}}$
$\ell=2$	$\{\boldsymbol{x}\in K\mapsto a\,\boldsymbol{x}+\boldsymbol{b},\ a\in\mathbb{R},\ \boldsymbol{b}\in\mathbb{R}^3\}$	$\mathbf{u}\mapsto\int_{\Sigma[\boldsymbol{a}_i,\boldsymbol{a}_j,\boldsymbol{a}_k]}\mathbf{u}\cdot\boldsymbol{n}\,dS$
$\ell=3$	$\{\boldsymbol{x}\mapsto c,\ c\in\mathbb{R}\}$	$u\mapsto\int_K u\,d\mathbf{x}$

Table 13.2.2.41: Vector proxy formulas for the spaces spanned by local Whitney ℓ-forms on K

$\triangleright$ $\beta^\ell_{f,K}\subset \mathcal{P}_1\Lambda^\ell(K)$

$\Rightarrow$ $d\beta^\ell_{f,K} = \text{const.}$

$\mathrm{Span}\{\beta^\ell_{f,K}\}_{f\in\mathcal{F}_\ell(K)} \neq \mathcal{P}_1\Lambda^\ell(K)$ for $\ell>0$

13.2.2.3 Global Whitney Forms

Definition 13.2.2.43. Generalized interpolation operator for cochains

Given an ℓ-cochain $\vec{\omega} \in \mathcal{C}^\ell(\mathcal{M})$, we define the generalized interpolation operators,

$$\mathsf{W}_\ell : \mathcal{C}^\ell(\mathcal{M}) \to \mathcal{I}^\ell(\Omega)\,, \quad 0 \le \ell \le d\,, \tag{13.2.2.5}$$

as

$$(\mathsf{W}_\ell\vec{\omega})(\boldsymbol{x}) := (\mathsf{W}_{\ell,K}\vec{\omega})(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x} \in K\,, \quad K \in \mathcal{F}_d(\mathcal{M})\,, \tag{13.2.2.44}$$

with $\mathsf{W}_{\ell,K}$ according to (13.2.2.17). In particular, $\mathsf{W}_\ell\vec{\omega}$ is a $\mathcal{M}$-piecwise smooth differential ℓ-form.

Issue: Well defined integral ℓ-form?

To check "tangential continuity"

Idea: $K \in \mathcal{F}_d(\mathcal{M})$, $F \in \mathcal{F}_{d-1}(K)$

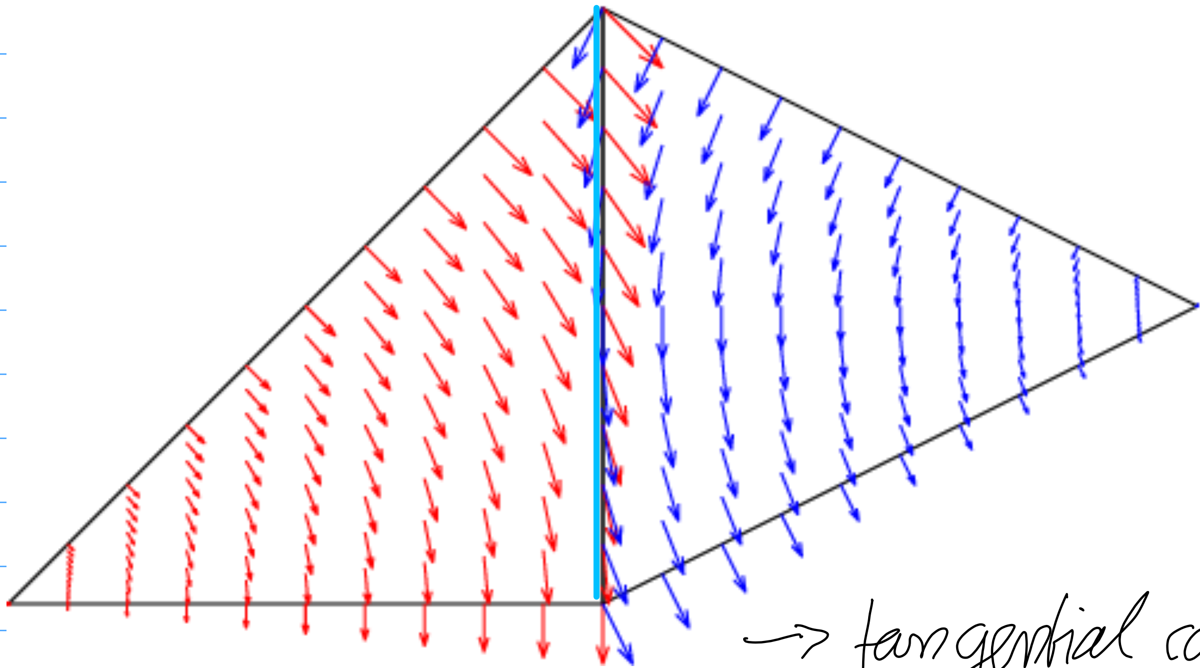
$f \in \mathcal{F}_\ell(K)$, $f \subset \overline{F}$

$\beta^\ell_{f,K|F}$ is "F-local": only barycentric coordinates of K associated w/ vertices of F will contribute ("the same from both sides")

$\triangleright$ $\mathsf{W}_\ell\vec{\omega} \in \mathcal{I}^\ell(\Omega)$

$\mathsf{W}_\ell\vec{\omega} \in H\Lambda^\ell(d,\Omega)$

2D, $\ell = 1$, VP (opt. I)



$\to$ tangential continuity

From $\mathsf{W}_{\ell,K}$:

$$d_\ell \circ \underline{\mathsf{W}}_\ell = \underline{\mathsf{W}}_{\ell+1} \circ \vec{d}_\ell$$

$\hookrightarrow$ ext. deriv. in $H\Lambda^\ell(d,\Omega)$

$$\langle \vec{\omega}_f, f \rangle = 1 \quad \text{and} \quad \langle \vec{\omega}_f, f' \rangle = 0 \quad \forall f' \in \mathcal{C}^\ell(\mathcal{M}) \setminus \{f\} \,.$$

We call the integral ℓ-form

$$\beta_f^\ell := \underline{\mathsf{W}}_\ell \vec{\omega}_f \in \mathcal{I}^\ell(\Omega) \tag{13.2.2.52}$$

the **Whitney ℓ-form** associated with the ℓ-facet f.

$\hookrightarrow$ A global shape function!

$$\operatorname{supp}(\beta_f^\ell) = \bigcup\{K \in \mathcal{F}_d(\mathcal{M}) : \ f \subset \overline{K}\} \,. \tag{13.2.2.53}$$

given a simplicial mesh $\mathcal{M}$ of Ω we have identified the **Whitney finite-element space** of degree $\ell \in \{0,\dots,d\}$,

$$\mathcal{W}^\ell(\mathcal{M}) := \underline{\mathsf{W}}_\ell(\mathcal{C}^\ell(\mathcal{M})) = \operatorname{span}\left(\left\{\beta_f^\ell\right\}_{f\in\mathcal{F}_\ell(\mathcal{M})}\right) \subset H\Lambda^\ell(\mathrm{d},\Omega)\,,$$

as a *finite-element subspace* of the Sobolev space $H\Lambda^\ell(\mathrm{d},\Omega)$ of (differential) ℓ-forms on Ω.

- p.w. linear
- Canonical basis $\{\beta_f^\ell\}_{f\in\mathcal{F}_\ell(\mathcal{M})}$
- $\mathcal{W}^0(\mathcal{M}) = \mathcal{S}_1^0(\mathcal{M})$

Subspace with "tangential" boundary values

$$\mathcal{W}_0^\ell(\mathcal{M}) = \mathcal{W}^\ell(\mathcal{M}) \cap H_0\Lambda^\ell(d,\Omega)$$
$$= \operatorname{span}\{\beta_f^\ell : f\in\mathcal{F}_\ell(\mathcal{M}),\ f \not\subset \partial\Omega\}$$

§ 13.2.2.58 : Review : Approximation properties of $\mathcal{W}^\ell(\mathcal{M})$

$\forall K \in \mathcal{F}_d(\mathcal{M})$:

$\mathcal{P}_0\Lambda^\ell(K) \subset \mathcal{W}^\ell(\mathcal{M})_{|K} \subset \mathcal{P}_1\Lambda^\ell(K)$

constant ℓ-forms

Heuristics, Ch. 3

If a finite-element space V_h locally contains all polynomials of degree p, then, for "sufficiently smooth" u,

$$\inf_{v_h \in V_h} \|u - v_h\|_{L^2(\Omega)} = O(h_{\mathcal{M}}^{p+1}) \quad \text{for} \quad h_{\mathcal{M}} \to 0\,,$$

and on uniformly shape regular meshes with meshwidths $h_{\mathcal{M}}$.

Here : $p = 0$

Theorem 13.2.2.60 L^2-norm best approximation estimates for $\mathcal{W}^\ell(\mathcal{M})$

If $\omega \in H^1\Lambda^\ell(\Omega)$ then

$$\inf_{v_h \in \mathcal{W}^\ell(\mathcal{M})} \|\omega - v_h\|_{L^2\Lambda^\ell(\Omega)} \leq C_\ell h_{\mathcal{M}} \|\omega\|_{H^1\Lambda^\ell(\Omega)}\,,$$

with a constant $C_\ell > 0$ that depends only on ℓ and the shape regularity measure (→ Def. 3.3.2.20) of $\mathcal{M}$.

first-order cvg.

Theorem 13.2.2.61. $H\Lambda^\ell(\mathrm{d},\Omega)$-norm best approximation estimates for $\mathcal{W}^\ell(\mathcal{M})$

If $\omega \in H^1\Lambda^\ell(\Omega)$ and $\mathrm{d}\omega \in H^1\Lambda^{\ell+1}(\Omega)$, then

$$\inf_{v_h \in \mathcal{W}^\ell(\mathcal{M})} \|\omega - v_h\|_{H\Lambda^\ell(\mathrm{d},\Omega)} \leq C_\ell h_{\mathcal{M}} \left(\|\omega\|_{H^1\Lambda^\ell(\Omega)} + \|\mathrm{d}_\ell\omega\|_{H^1\Lambda^{\ell+1}(\Omega)}\right)\,,$$

with a constant $C_\ell > 0$ that depends only on ℓ and the shape regularity measure (→ Def. 3.3.2.20) of $\mathcal{M}$.

▷ Review questions 13.2.2.7.8