ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 13.3.2.1: Electromagnetic Wave Equations

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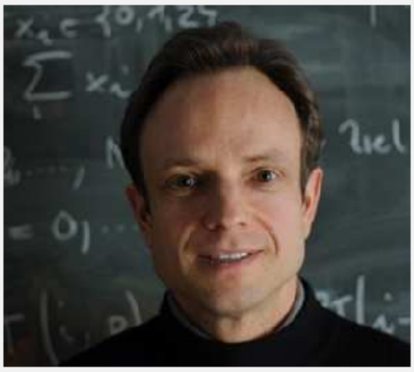

Prerequisites.

- Calculus and ODEs

Dependencies. [Lecture → Section 13.1.4], [Lecture → Section 13.1.5], [Lecture → Section 13.3.1.2]

Duration: minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XIII Finite-Element Exterior Calculus (FEEC)

13.3. Whitney FEM for Computational Electromagnetism

13.3.2. Electrodynamic Evolution Problems

IBVP for EM fields

ME: $\mathsf{d}_1\mathbf{e}(t) = -\frac{d}{dt}\mathbf{b}(t)$ in $L^2\Lambda^2(\Omega)$, $\mathsf{d}_1\mathbf{h}(t) = \frac{d}{dt}\mathbf{d}(t)+\mathbf{j}(t)$ in $L^2\Lambda^2(\Omega)$, (13.3.1.1)

ML:

$$\boxed{\begin{aligned}\int_\Omega \mathbf{d}(t)\wedge\mathbf{e}' &= \mathsf{m}_\epsilon(\mathbf{e}(t),\mathbf{e}') \quad \forall \mathbf{e}'\in L^2\Lambda^1(\Omega)\,, & (13.3.1.22)\\ \int_\Omega \mathbf{b}(t)\wedge\mathbf{h}' &= \mathsf{m}_\mu(\mathbf{h}(t),\mathbf{h}') \quad \forall \mathbf{h}'\in L^2\Lambda^1(\Omega)\,. & (13.3.1.23)\end{aligned}}$$

b.c.: (artificial / idealization)

In v.p.: $\check{\mathbf{e}}(t)\times\boldsymbol{n}=0$ [PEC, (13.1.3.43)] **or** $\check{\mathbf{h}}(t)\times\boldsymbol{n}=0$ [PMC, (13.1.3.44)] on $\partial\Omega$. (13.3.2.1)

$\hat{=}$ zero tangential b.c. for 1-forms

13.3.2.1 Electromagnetic Wave Equations

Goal: spatial variational formulation, PEC b.c.

$\frac{d}{dt}$ on $\mathcal{M}$:

$$m_\epsilon(\dot{e}(t), e') = \int_\Omega \dot{d}(t)\wedge e' \overset{ME}{=} \int_\Omega (d_1 h(t) - j)\wedge e' \qquad (I)$$

$$= \int_\Omega h(t)\wedge d_1 e' - j(t)\wedge e' + \overset{=0}{\cancel{\int_{\partial\Omega} h(t)\wedge e'}}$$

[because of zero tangential b.c. on both $e(t)$ and e']

$$m_\mu(\dot{h}(t), h') = \int \dot{b}(t)\wedge h' = \int_\Omega -d_1 e(t)\wedge h' \qquad (II)$$

for $e, h : [0,T] \to H\Lambda^1(d,\Omega)$ $\quad \forall e', h' \in L^2\Lambda^1(\Omega)$

mismatch of test/trial spaces

Apply integration by parts (13.1.4.124) (for $\ell = k = 1$) to *one* equation of (I), (II).

PEC b.c. are essential b.c.

Seek $\mathbf{e} : [0,T] \to H_0\Lambda^1(\mathrm{d},\Omega)$ and $\mathbf{h} : [0,T] \to L^2\Lambda^1(\Omega)$ such that

$$\begin{aligned} \mathrm{m}_\epsilon\left(\frac{d\mathbf{e}}{dt}(t), \mathbf{e}'\right) &- \int_\Omega \mathbf{h}(t)\wedge \mathrm{d}_1\mathbf{e}' = -\int_\Omega \mathbf{j}(t)\wedge\mathbf{e}' \quad \forall \mathbf{e}' \in H_0\Lambda^1(\mathrm{d},\Omega), \\ \mathrm{m}_\mu\left(\frac{d\mathbf{h}}{dt}(t), \mathbf{h}'\right) &+ \int_\Omega \mathrm{d}_1\mathbf{e}(t)\wedge\mathbf{h}' = 0 \quad \forall \mathbf{h}' \in L^2\Lambda^1(\Omega)\,. \end{aligned} \qquad (13.3.2.6)$$

$\hat{=}$ 1st-order "ODE in function space"

$\Rightarrow$ Requires initial conditions

$e(0) = e_0 \in H_0\Lambda^1(d,\Omega)$, $h(0) = h_0 \in L^2\Lambda^1(\Omega)$

§ 13.3.2.13 Conservation of energy

$$\begin{aligned}
&\mathsf{m}_\epsilon(\frac{d\mathbf{e}}{dt}(t),\mathbf{e}') &- \quad& \int_\Omega \mathbf{h}(t)\wedge \mathsf{d}_1\mathbf{e}' &=\quad& -\int_\Omega \mathbf{j}(t)\wedge\mathbf{e}' \quad \forall \mathbf{e}' \in H_0\Lambda^1(\mathsf{d},\Omega)\,,\\
+\;&\mathsf{m}_\mu(\frac{d\mathbf{h}}{dt}(t),\mathbf{h}') &+ \quad& \int_\Omega \mathsf{d}_1\mathbf{e}(t)\wedge\mathbf{h}' &=\quad& 0 \qquad\qquad\quad\; \forall \mathbf{h}' \in L^2\Lambda^1(\Omega)\,,
\end{aligned} \tag{13.3.2.6}$$

Set $e' \leftarrow e(t)$, $h' \leftarrow h(t)$ & add equ.

$$\mathsf{m}_\epsilon(\frac{d\mathbf{e}}{dt}(t),\mathbf{e}(t)) + \mathsf{m}_\mu(\frac{d\mathbf{h}}{dt}(t),\mathbf{h}(t)) = -\int_\Omega \mathbf{j}(t)\wedge\mathbf{e}(t)\,. \tag{13.3.2.14}$$

$$E_{\text{tot}}(t) = E_{\text{el}}(\mathbf{e}(t)) + E_{\text{mag}}(\mathbf{h}(t)) = \tfrac{1}{2}\mathsf{m}_\epsilon(\mathbf{e}(t),\mathbf{e}(t)) + \tfrac{1}{2}\mathsf{m}_\mu(\mathbf{h}(t)),\mathbf{h}(t))\,.$$

$$(14) \Rightarrow \frac{d}{dt}E_{tot}(t) = \underbrace{-\int_\Omega j(t)\wedge e(t)}_{\text{power injected by source current}}$$

$\leftrightarrow$ Poynting's thm.

§ 13.3.2.19 Constraint propagation

Seek $\mathbf{e} : [0,T] \to H_0\Lambda^1(\mathsf{d},\Omega)$ and $\mathbf{h} : [0,T] \to L^2\Lambda^1(\Omega)$ such that

$$
\begin{aligned}
\mathsf{m}_\epsilon\left(\frac{d\mathbf{e}}{dt}(t),\mathbf{e}'\right) &- \int_\Omega \mathbf{h}(t)\wedge \mathsf{d}_1\mathbf{e}' &= -\int_\Omega \mathbf{j}(t)\wedge\mathbf{e}' \quad &\forall \mathbf{e}' \in H_0\Lambda^1(\mathsf{d},\Omega)\,,\\
\mathsf{m}_\mu\left(\frac{d\mathbf{h}}{dt}(t),\mathbf{h}'\right) &+ \int_\Omega \mathsf{d}_1\mathbf{e}(t)\wedge\mathbf{h}' &= 0 \quad &\forall \mathbf{h}' \in L^2\Lambda^1(\Omega)\,.
\end{aligned}
\tag{13.3.2.6}
$$

Use $\frac{d}{dt}$ 'ML : $\int_\Omega b(t)\wedge h' = m_\mu(h(t),h')$

$\longrightarrow \int_\Omega \dot{b}(t)\wedge h' + \int_\Omega d_1 e(t)\wedge h' = 0 \quad \forall h'$

test $h' := d_0 v'$, $v \in H\Lambda^0(d,\Omega)$

$$\int_\Omega \dot{b}(t)\wedge d_0 v' + \underbrace{\int_\Omega d_1 e(t)\wedge d_0 v'}_{=0 \text{ by i.b.p } \& \ e(t)|_{\partial\Omega}=0} = 0$$

$$\Big\downarrow$$

$$-\int_\Omega d_2\dot{b}(t)\wedge v' + \int_{\partial\Omega} \dot{b}(t)\wedge v' = 0 \quad \forall v'$$

$$\Downarrow \qquad\qquad \Downarrow$$

$\frac{d}{dt} d_2 b(t) = 0$ in Ω, $\quad b|_{\partial\Omega} = 0$ on $\partial\Omega$

[tangential b.c.]

V.P. $\frac{d}{dt}\operatorname{div}\check{b}(t) = 0 \qquad \check{b}(t)\cdot n = 0$ on $\partial\Omega$

Seek $\mathbf{e} : [0,T] \to H_0\Lambda^1(\mathrm{d},\Omega)$ and $\mathbf{h} : [0,T] \to L^2\Lambda^1(\Omega)$ such that

$$\begin{aligned} \mathrm{m}_\epsilon\left(\frac{d\mathbf{e}}{dt}(t),\mathbf{e}'\right) &- \int_\Omega \mathbf{h}(t)\wedge \mathrm{d}_1\mathbf{e}' = -\int_\Omega \mathbf{j}(t)\wedge\mathbf{e}' \quad \forall \mathbf{e}' \in H_0\Lambda^1(\mathrm{d},\Omega)\,, \\ \mathrm{m}_\mu\left(\frac{d\mathbf{h}}{dt}(t),\mathbf{h}'\right) &+ \int_\Omega \mathrm{d}_1\mathbf{e}(t)\wedge\mathbf{h}' = 0 \quad \forall \mathbf{h}' \in L^2\Lambda^1(\Omega)\,. \end{aligned} \tag{13.3.2.6}$$

Use $\frac{d}{dt}$ ML : $\int_\Omega \dot{d}(t)\wedge e' = m_\varepsilon(\dot{e}(t),e')$

$\Rightarrow \int_\Omega \dot{d}(t)\wedge e' - \int_\Omega h(t)\wedge d_1 e' = -\int_\Omega j(t)\wedge e' \quad \forall e'$

Use: $e' := dv'$, $v' \in H_0\Lambda^0(d,\Omega)$

$$\Rightarrow \int_\Omega \dot{d}(t)\wedge dv' = -\int_\Omega j(t)\wedge d_0 v'$$

$$\Rightarrow -\int_\Omega d_2\dot{d}(t)\wedge v' = \int_\Omega d_2 j(t)\wedge v'$$

$$\Rightarrow \text{PDE} \quad -\frac{d}{dt}\underbrace{d_2 d(t)}_{=:\, q \,\hat{=}\, \text{charge density 3-form}} = d_2 j(t)$$

$\Rightarrow$ Continuity equation

$$-\frac{d}{dt}q(t) = d_2 j(t)$$

$$\text{V.P.:} \quad \frac{\partial \check{q}}{\partial t}(x,t) = -\operatorname{div}\check{j}(x,t)$$

§ 13.3.2.31 EMWE in 2nd-order form

(for local linear ML, v.p. notation]

$\check{e}:]0,T[\to H_0(\mathrm{curl},\Omega)$, $\check{h}:]0,T[\to (L^2(\Omega))^3$

$$\begin{aligned}
\int_\Omega \left(\epsilon(x)\frac{\partial \check{\mathbf{e}}}{\partial t}(x,t)\right)\cdot \check{\mathbf{e}}'(x)\,\mathrm{d}x - \int_\Omega \check{\mathbf{h}}(x,t)\cdot \mathbf{curl}\,\check{\mathbf{e}}'(x)\,\mathrm{d}x &= -\int_\Omega \check{\mathbf{j}}(x,t)\cdot\check{\mathbf{e}}'(x)\,\mathrm{d}x\,,\\
\int_\Omega \left(\mu(x)\frac{\partial \check{\mathbf{h}}}{\partial t}(x,t)\right)\cdot \check{\mathbf{h}}'(x)\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\check{\mathbf{e}}(x,t)\cdot\check{\mathbf{h}}'(x)\,\mathrm{d}x &= 0
\end{aligned} \tag{13.3.2.7}$$

for all $\forall \check{\mathbf{e}}' \in \boldsymbol{H}_0(\mathbf{curl},\Omega)$, $\check{\mathbf{h}}' \in (L^2(\Omega))^3$.

In 2nd equ.: $h' \leftarrow \mu^{-1} b'$

$$\begin{aligned}
\frac{d}{dt}\Big|\quad \int_\Omega \left(\epsilon(x)\frac{\partial \check{\mathbf{e}}}{\partial t}(x,t)\right)\cdot \check{\mathbf{e}}'(x)\,\mathrm{d}x - \int_\Omega \check{\mathbf{h}}(x,t)\cdot \mathbf{curl}\,\check{\mathbf{e}}'(x)\,\mathrm{d}x &= -\int_\Omega \check{\mathbf{j}}(x,t)\cdot\check{\mathbf{e}}'(x)\,\mathrm{d}x\,,\\
\int_\Omega \frac{\partial \check{\mathbf{h}}}{\partial t}(x,t)\cdot \check{\mathbf{b}}'(x)\,\mathrm{d}x + \int_\Omega \left(\mu^{-1}(x)\,\mathbf{curl}\,\check{\mathbf{e}}(x,t)\right)\cdot\check{\mathbf{b}}'(x)\,\mathrm{d}x &= 0
\end{aligned}$$

for all $\forall \check{\mathbf{e}}' \in \boldsymbol{H}_0(\mathbf{curl},\Omega)$, $\check{\mathbf{h}}' \in (L^2(\Omega))^3$.

$\check{b}' := \mathrm{curl}\,\check{e}'$

$$\begin{aligned}
\int_\Omega \left(\epsilon(x)\frac{\partial^2 \check{\mathbf{e}}}{\partial t^2}(x,t)\right)\cdot \check{\mathbf{e}}'(x)\,\mathrm{d}x - \int_\Omega \frac{\partial \check{\mathbf{h}}}{\partial t}(x,t)\cdot \mathbf{curl}\,\check{\mathbf{e}}'(x)\,\mathrm{d}x &= -\int_\Omega \frac{\partial \check{\mathbf{j}}}{\partial t}(x,t)\cdot\check{\mathbf{e}}'(x)\,\mathrm{d}x\,,\\
\int_\Omega \frac{\partial \check{\mathbf{h}}}{\partial t}(x,t)\cdot \mathbf{curl}\,\check{\mathbf{e}}'(x)\,\mathrm{d}x + \int_\Omega \left(\mu^{-1}(x)\,\mathbf{curl}\,\check{\mathbf{e}}(x,t)\right)\cdot\mathbf{curl}\,\check{\mathbf{e}}'(x)\,\mathrm{d}x &= 0
\end{aligned}$$

+ for all $\forall \check{\mathbf{e}}' \in \boldsymbol{H}_0(\mathbf{curl},\Omega)$.

Seek $\mathbf{e} :]0,T[\to \boldsymbol{H}_0(\mathbf{curl},\Omega)$ such that

$$\boxed{\int_\Omega \left(\epsilon(x)\frac{d^2 \check{\mathbf{e}}}{dt^2}(x,t)\right)\cdot \check{\mathbf{e}}'(x)\,\mathrm{d}x + \int_\Omega \left(\mu^{-1}(x)\,\mathbf{curl}\,\check{\mathbf{e}}(x,t)\right)\cdot\mathbf{curl}\,\check{\mathbf{e}}'(x)\,\mathrm{d}x = -\int_\Omega \frac{\partial \check{\mathbf{j}}}{\partial t}(x,t)\cdot\check{\mathbf{e}}'(x)\,\mathrm{d}x} \tag{13.3.2.32}$$

for all $\check{\mathbf{e}}' \in \boldsymbol{H}_0(\mathbf{curl},\Omega)$.

Recall Sect 9.2 :

$$u(t) \in V_0\colon\ \mathsf{m}\left(\frac{d^2u}{dt^2}, v\right) + \mathsf{a}(u,v) = \ell(v) \quad \forall v \in V_0\,. \tag{9.3.1.8}$$

with

- function space $V_0 := \boldsymbol{H}_0(\mathbf{curl}, \Omega)$,
- an s.p.d. "L^2-type" bilinear form $\mathsf{m} = \mathsf{m}_\epsilon$, $\quad \mathsf{m}_\epsilon(\check{\mathbf{e}}, \check{\mathbf{e}}') := \int_\Omega (\epsilon(\boldsymbol{x})\check{\mathbf{e}}(\boldsymbol{x})) \cdot \check{\mathbf{e}}'(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$,
- the bilinear form corresponding to a second-order PDE operator in space

$$\mathsf{a} \quad \leftrightarrow \quad (\check{\mathbf{e}}, \check{\mathbf{e}}') \mapsto \int_\Omega (\mu^{-1}(\boldsymbol{x})\, \mathbf{curl}\, \check{\mathbf{e}}(\boldsymbol{x},t)) \cdot \mathbf{curl}\, \check{\mathbf{e}}'(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\,,$$

- and a right-hand side linear form

$$\ell \quad \leftrightarrow \quad \mathbf{e}' \mapsto -\int_\Omega \check{\mathbf{j}}(\boldsymbol{x},t) \cdot \check{\mathbf{e}}'(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\,.$$

EMWE share key features with scalar WE

$\rightarrow$ finite speed of propagation $\leq s_{max}$

$\mathrm{supp}\, e_0, h_0 \subset \Omega_0$:

$\mathrm{supp}\, e(t), \mathrm{supp}\, h(t) \subset \{x \in \mathbb{R}^3 : \mathrm{dist}(\Omega, x) \leq t \cdot s_{max}\}$

$\rightarrow$ domain of influence & dependence