ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 13.3.2.2–Section 13.3.2.3: Method of lines for Electromagnetic Wave Equations, Timestepping for Semi-Discrete Electromagnetic Wave Equation

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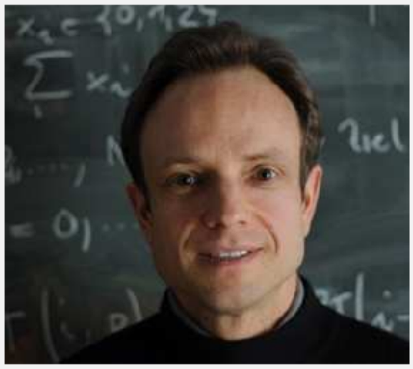


Prerequisites.

- Elementary linear algebra and difference quotients

Dependencies. [Lecture → Section 9.3.3], [Lecture → Section 9.3.4], [Lecture → Section 9.3.5], [Lecture → Section 13.3.2.1], [Lecture → Section 13.2.2.3]

Duration: 50 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XIII Finite-Element Exterior Calculus (FEEC)

13.3. Whitney FEM for Computational Electromagnetism

13.3.2. Electrodynamic Evolution Problems

13.3.2.2. Method of Lines for Electromagnetic Wave Equations

Starting point:

Seek $\mathbf{e} : [0,T] \to H_0\Lambda^1(\mathrm{d},\Omega)$ and $\mathbf{h} : [0,T] \to L^2\Lambda^1(\Omega)$ such that

$$\begin{aligned} \mathsf{m}_\epsilon\left(\frac{d\mathbf{e}}{dt}(t),\mathbf{e}'\right) &- \int_\Omega \mathbf{h}(t)\wedge \mathrm{d}_1\mathbf{e}' &= -\int_\Omega \mathbf{j}(t)\wedge\mathbf{e}' \quad &\forall \mathbf{e}'\in H_0\Lambda^1(\mathrm{d},\Omega)\,,\\ \mathsf{m}_\mu\left(\frac{d\mathbf{h}}{dt}(t),\mathbf{h}'\right) &+ \int_\Omega \mathrm{d}_1\mathbf{e}(t)\wedge\mathbf{h}' &= 0 \quad &\forall \mathbf{h}'\in L^2\Lambda^1(\Omega)\,. \end{aligned} \tag{13.3.2.6}$$

MOL: Stage 1: Spatial Galerkin discretization

Step I: Trial/test spaces → f.d. subspaces

$\mathcal{M} \hat{=}$ simplicial mesh of Ω

For $H_0\Lambda^1(d,\Omega)$ use $\mathcal{W}_0^1(\mathcal{M})$

For $L^2\Lambda^1(\Omega)$ use $\mathcal{P}_0\Lambda^1(\mathcal{M})$

$\uparrow$ $\mathcal{M}$-p.w. constant 1-forms

Seek $\mathbf{e}_h : [0,T] \to \mathcal{W}_0^1(\mathcal{M})$ and $\mathbf{h}_h : [0,T] \to \mathcal{P}_0\Lambda^1(\mathcal{M})$ such that

$$\begin{aligned} \mathsf{m}_\epsilon\left(\frac{d\mathbf{e}_h}{dt}(t),\mathbf{e}_h'\right) - \int_\Omega \mathbf{h}_h(t) \wedge \mathsf{d}_1\mathbf{e}_h' &= -\int_\Omega \mathbf{j}(t) \wedge \mathbf{e}_h' \quad \forall \mathbf{e}_h' \in \mathcal{W}_0^1(\mathcal{M}), \\ \mathsf{m}_\mu\left(\frac{d\mathbf{h}_h}{dt}(t),\mathbf{h}_h'\right) + \int_\Omega \mathsf{d}_1\mathbf{e}_h(t) \wedge \mathbf{h}_h' &= 0 \quad \forall \mathbf{h}_h' \in \cancel{L^2\Lambda^1(\Omega)}\ \mathcal{P}_0\Lambda^1(\mathcal{M}). \end{aligned} \tag{13.3.2.35}$$

Step II: Choose ordered bases

- $\mathcal{W}_0^1(\mathcal{M})$: use Whitney 1-forms
- $\mathcal{P}_0\Lambda^1(\mathcal{M})$: 3 basis 1-forms per cell, v.p. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\triangleright$ MOL-ODE

Seek time-dependent basis expansion coefficient vectors $\vec{\mathbf{e}} : [0,T] \to \mathbb{R}^n$ and $\vec{\mathbf{h}} : [0,T] \to \mathbb{R}^m$ such that

$$\begin{aligned} \mathbf{M}_\epsilon \frac{d\vec{\mathbf{e}}}{dt}(t) - \mathbf{B}^\top \vec{\mathbf{h}}(t) &= \vec{\boldsymbol{\psi}}(t), \\ \mathbf{M}_\mu \frac{d\vec{\mathbf{h}}}{dt}(t) + \mathbf{B}\vec{\mathbf{e}}(t) &= 0, \end{aligned} \tag{13.3.2.36}$$

$\vec{e}(t), \ldots$ $\hat{=}$ time-dependent basis expansion vectors

§ 13.3.2.37 MOL: energy balance

Seek $\mathbf{e}_h : [0,T] \to \mathcal{W}_0^1(\mathcal{M})$ and $\mathbf{h}_h : [0,T] \to \mathcal{P}_0\Lambda^1(\mathcal{M})$ such that

$$\begin{aligned}
\mathsf{m}_\epsilon\left(\frac{d\mathbf{e}_h}{dt}(t), \mathbf{e}_h'\right) - \int_\Omega \mathbf{h}_h(t) \wedge \mathsf{d}_1\mathbf{e}_h' &= -\int_\Omega \mathbf{j}(t) \wedge \mathbf{e}_h' \quad \forall \mathbf{e}_h' \in \mathcal{W}_0^1(\mathcal{M}), \\
\mathsf{m}_\mu\left(\frac{d\mathbf{h}_h}{dt}(t), \mathbf{h}_h'\right) + \int_\Omega \mathsf{d}_1\mathbf{e}_h(t) \wedge \mathbf{h}_h' &= 0 \quad \forall \mathbf{h}_h' \in \cancel{L^2\Lambda^1(\Omega)}\ \mathcal{S}_0\Lambda^1(\mathcal{M}).
\end{aligned} \tag{13.3.2.35}$$

- $e_h' \leftarrow e_h(t)$, $h_h' \leftarrow h_h(t)$
- i.b.p. & add ($\longrightarrow$ cancellation)

$$\frac{dE_{\mathrm{tot},h}}{dt}(t) = -\int_\Omega \mathbf{j}(t) \wedge \mathbf{e}_h(t), \quad E_{\mathrm{tot},h} := \tfrac{1}{2}\mathsf{m}_\epsilon(\mathbf{e}_h(t), \mathbf{e}_h(t)) + \tfrac{1}{2}\mathsf{m}_\mu(\mathbf{h}_h(t), \mathbf{h}_h(t)), \tag{13.3.2.38}$$

$\updownarrow$

$$\boxed{\frac{dE_{\mathrm{tot}}}{dt}(t) = -\int_\Omega \mathbf{j}(t) \wedge \mathbf{e}(t)}\ . \tag{13.3.2.15}$$

§ 13.3.2.39 MOL: Constraint propagation

Recall:

$\frac{d}{dt}\mathsf{d}_2 b(t) = 0$

$$\int_\Omega \mathbf{b}(t) \wedge \mathsf{d}_0 v' = \int_\Omega \mathbf{b}_0 \wedge \mathsf{d}_0 v' \quad \forall v' \in H\Lambda^0(\mathsf{d}, \Omega) \text{ and all times } t, \tag{13.3.2.21}$$

$$\int_\Omega \frac{d\mathbf{d}}{dt}(t) \wedge \mathsf{d}_0 v' = -\int_\Omega \mathbf{j}(t) \wedge \mathsf{d}_0 v' \quad \forall v' \in H_0\Lambda^0(\mathsf{d}, \Omega) \text{ and for all times } t. \tag{13.3.2.27}$$

$\longrightarrow$ continuity equ.

Observe:

- $\mathsf{d}_0 \mathcal{W}_0^0(\mathcal{M}) \subset \mathcal{W}_0^1(\mathcal{M})$
- $\mathsf{d}_0 \mathcal{W}^0(\mathcal{M}) \subset \mathcal{S}_0\Lambda^1(\mathcal{M})$

Seek $\mathbf{e}_h : [0,T] \to \mathcal{W}_0^1(\mathcal{M})$ and $\mathbf{h}_h : [0,T] \to \mathcal{P}_0\Lambda^1(\mathcal{M})$ such that

$$\begin{aligned} \mathsf{m}_\epsilon(\frac{d\mathbf{e}_h}{dt}(t),\mathbf{e}_h') \quad &- \quad \int_\Omega \mathbf{h}_h(t)\wedge \mathsf{d}_1\mathbf{e}_h' \quad = \quad -\int_\Omega \mathbf{j}(t)\wedge \mathbf{e}_h' \quad \forall \mathbf{e}_h' \in \mathcal{W}_0^1(\mathcal{M})\,, \\ \mathsf{m}_\mu(\frac{d\mathbf{h}_h}{dt}(t),\mathbf{h}_h') \quad &+ \quad \int_\Omega \mathsf{d}_1\mathbf{e}_h(t)\wedge \mathbf{h}_h' \quad = \quad 0 \qquad \forall \mathbf{h}_h' \in \cancel{L^2\Lambda^1(\Omega)}\ \mathcal{P}_0\Lambda^1(\mathcal{M})\,. \end{aligned} \tag{13.3.2.35}$$

$$e_h' \longleftarrow d_0 v_h\,,\quad v_h \in W_0^0(\mathcal{M})$$

$$h_h' \longleftarrow d_0 w_h\,,\quad w_h \in W^0(\mathcal{M})$$

By i.b.p: $\displaystyle\int_\Omega d_1 e_h(t) \wedge d_0 v_h = 0$

$\triangleright$

$$\mathsf{m}_\epsilon(\frac{d\mathbf{e}_h}{dt}(t),\mathsf{d}_0 v_h) = -\int_\Omega \mathbf{j}(t)\wedge \mathsf{d}_0 v_h \qquad \forall v_h \in \mathcal{W}_0^0(\mathcal{M})\,, \tag{13.3.2.40}$$

$$\mathsf{m}_\mu(\frac{d\mathbf{h}_h}{dt}(t),\mathsf{d}_0 w_h) = 0 \qquad \forall w_h \in \mathcal{W}^0(\mathcal{M})\,. \tag{13.3.2.41}$$

For local, linear ML & v.p. $\Updownarrow$

$\approx \check{\underline{d}}(x,t)$

$$\int_\Omega \left(\frac{\partial}{\partial t}(\epsilon(\mathbf{x})\check{\mathbf{e}}_h(\mathbf{x},t))\right)\cdot \mathbf{grad}\,\check{v}_h(\mathbf{x})\,\mathrm{d}\mathbf{x} = -\int_\Omega \check{\mathbf{j}}(\mathbf{x},t)\cdot \mathbf{grad}\,\check{v}_h(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \forall \check{v}_h \in \mathcal{S}_{1,0}^0(\mathcal{M})\,, \tag{13.3.2.43}$$

$$\int_\Omega \left(\frac{\partial}{\partial t}(\mu(\mathbf{x})\check{\mathbf{h}}_h(\mathbf{x},t))\right)\cdot \mathbf{grad}\,\check{v}_h(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \quad \forall v_h \in \mathcal{S}_1^0(\mathcal{M})\,. \tag{13.3.2.44}$$

$\check{\underline{b}}(x,t)$ → variational way to express $\operatorname{div} \check{b} = 0$

No i.b.p. possible, because red terms lack smoothness for evaluation of div!

13.3.2.3 Timestepping for semi-discrete EMWE

1st-order linear ODE in $\mathbb{R}^N$

$$
\begin{aligned}
\mathbf{M}_\epsilon \frac{d\vec{\mathbf{e}}}{dt}(t) &- \mathbf{B}^\top \vec{\mathbf{h}}(t) &= \vec{\psi}(t), \\
\mathbf{M}_\mu \frac{d\vec{\mathbf{h}}}{dt}(t) &+ \mathbf{B}\vec{\mathbf{e}}(t) &= 0,
\end{aligned}
\qquad (13.3.2.36)
$$

MOL stage II: timestepping → Sect. 9.3.4

Temporal grid $0 = t_0 < t_1 < \dots < t_M = T$, $T > 0$ final time

§ 13.3.2.45 Crank-Nicolson t.s.

$\leftrightarrow$ RK-SSM $\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$ $\hat{=}$ implicit midpoint SSM

$$
\begin{aligned}
\mathbf{M}_\epsilon \left(\frac{\vec{\mathbf{e}}^{(j)} - \vec{\mathbf{e}}^{(j-1)}}{\tau_j} \right) &- \mathbf{B}^\top \left(\frac{\vec{\mathbf{h}}^{(j)} + \vec{\mathbf{h}}^{(j-1)}}{2} \right) &= \vec{\psi}(t_{j-\frac{1}{2}}), \\
\mathbf{M}_\mu \left(\frac{\vec{\mathbf{h}}^{(j)} - \vec{\mathbf{h}}^{(j-1)}}{\tau_j} \right) &+ \mathbf{B} \left(\frac{\vec{\mathbf{e}}^{(j)} + \vec{\mathbf{e}}^{(j-1)}}{2} \right) &= 0,
\end{aligned}
\qquad j = 1, \dots, M, \qquad (13.3.2.46)
$$

$$
\begin{bmatrix} \mathbf{M}_\epsilon & -\frac{1}{2}\tau_j \mathbf{B}^\top \\ \frac{1}{2}\tau_j \mathbf{B} & \mathbf{M}_\mu \end{bmatrix}
\begin{bmatrix} \vec{\mathbf{e}}^{(j)} \\ \vec{\mathbf{h}}^{(j)} \end{bmatrix}
=
\begin{bmatrix} \mathbf{M}_\epsilon & \tau_j \frac{1}{2} \mathbf{B}^\top \\ -\frac{1}{2}\tau_j \mathbf{B} & \mathbf{M}_\mu \end{bmatrix}
\begin{bmatrix} \vec{\mathbf{e}}^{(j-1)} \\ \vec{\mathbf{h}}^{(j-1)} \end{bmatrix}
+
\begin{bmatrix} \vec{\psi}(t_{j-\frac{1}{2}}) \\ \mathbf{0} \end{bmatrix}. \qquad (13.3.2.51)
$$

→ Fully implicit method

▷ Discrete energy balance

$$E_{\mathrm{tot}}^{(j)} = E_{\mathrm{tot}}^{(j-1)} + \tau_j \tfrac{1}{2}\left(\vec{\mathbf{e}}^{(j)} + \vec{\mathbf{e}}^{(j-1)}\right)^\top \vec{\boldsymbol{\psi}}(t_{j-\frac{1}{2}}), \qquad (13.3.2.47)$$

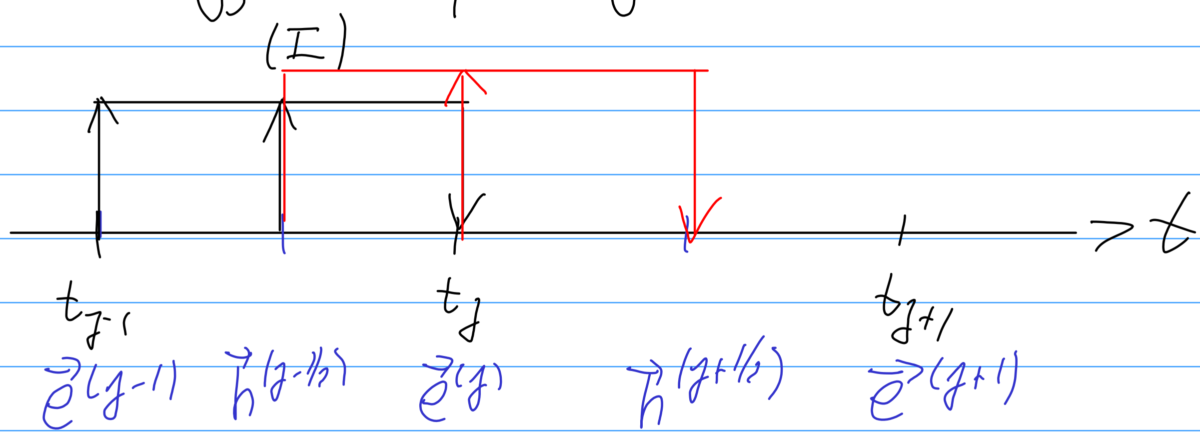
$$E_{\mathrm{tot}}^{(k)} := \tfrac{1}{2}\left(\vec{\mathbf{e}}^{(k)}\right)^\top \mathbf{M}_\epsilon \vec{\mathbf{e}}^{(k)} + \tfrac{1}{2}\left(\vec{\mathbf{h}}^{(k)}\right)^\top \mathbf{M}_\mu \vec{\mathbf{h}}^{(k)},$$

$$= \tfrac{1}{2} m_\epsilon(e_h^{(k)}, e_h^{(k)}) + \tfrac{1}{2} m_\mu(h_h^{(k)}, h_h^{(k)})$$

Compare: $$\frac{dE_{\mathrm{tot}}}{dt}(t) = -\int_\Omega \mathbf{j}(t) \wedge \mathbf{e}(t) \,. \qquad (13.3.2.15)$$

§ 13.3.2.48 Leapfrog timestepping

→ Use staggered temporal grid



$$\begin{aligned}
\mathbf{M}_\epsilon \left(\frac{\vec{\mathbf{e}}^{(j)} - \vec{\mathbf{e}}^{(j-1)}}{\tau}\right) &- \mathbf{B}^\top \vec{\mathbf{h}}^{(j-\frac{1}{2})} = \vec{\boldsymbol{\psi}}(t_{j-\frac{1}{2}}), \quad j = 1,\dots,M, && (\mathrm{I}) \\
\mathbf{M}_\mu \left(\frac{\vec{\mathbf{h}}^{(j+\frac{1}{2})} - \vec{\mathbf{h}}^{(j-\frac{1}{2})}}{\tau}\right) &+ \mathbf{B}\,\vec{\mathbf{e}}^{(j)} = 0, \quad j = 0,\dots,M-1. && (\mathrm{II})
\end{aligned} \qquad (13.3.2.49)$$

1. $$\mathbf{M}_\epsilon \vec{\mathbf{e}}^{(j)} = \mathbf{M}_\epsilon \vec{\mathbf{e}}^{(j-1)} + \tau \mathbf{B}^\top \vec{\mathbf{h}}^{(j-\frac{1}{2})} + \tau \vec{\boldsymbol{\psi}}(t_{j-\frac{1}{2}}),$$

2. $$\mathbf{M}_\mu \vec{\mathbf{h}}^{(j+\frac{1}{2})} = \mathbf{M}_\mu \vec{\mathbf{h}}^{(j-\frac{1}{2})} - \tau \mathbf{B}\,\vec{\mathbf{e}}^{(j)} .$$

Not explicit : Have to solve LSEs w/ M_ε, M_μ

Scalar WE & $S_1^0(\mathcal{M})$ $\Longrightarrow$ Mass lumping*
(Approximate M_ε, M_μ by diagonal matrices ?)

* Not available in general for Whitney FE