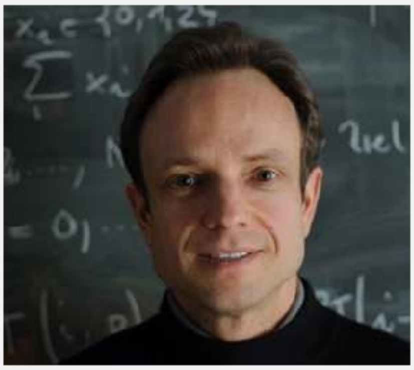
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 13.3.3.1: Magnetostatics: Formulations Based on Scalar Potentials

Prof. R. Hiptmair, SAM, ETH Zurich

Date: May 9, 2025



Dependencies. [Lecture → Section 1.8], [Lecture → Section 13.3.1.2], [Lecture → Section 13.1.4], [Lecture → Section 13.1.5].

Duration: 30 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XIII Finite-Element Exterior Calculus (FEEC)

13.3. Whitney FEM for Computational Electromagnetism

13.3.3. Magnetostatics

Static setting: fields do not depend on time, $\frac{d}{dt}=0$

↳ ME:

constraints in transient setting

↓

electrostatics: (FL): $d_1\mathbf{e}=0$, $d_2\mathbf{d}=q$, (13.3.3.3)

magnetostatics: (AL): $d_1\mathbf{h}=\mathbf{j}$, $d_2\mathbf{b}=0$. (13.3.3.4)

↳ necessary: $d_2 j = 0$

+ demand: $\mathrm{supp}(j) \Subset \Omega \Rightarrow j_{|\partial\Omega} = 0$

[v.p.: $\check{j}\cdot n = 0$ on $\partial\Omega$]

v.p.

$\mathbf{curl}\,\check{h} = \check{j}$ $\qquad \check{b} = \mu(x)\check{h}$ (ML) $\qquad \mathrm{div}\,\check{b} = 0$

b.c.: (artificial truncation $\mathbb{R}^3 \to \Omega$, PMC)

$\mathbf{h} \in H_0\Lambda^1(\mathrm{d},\Omega)$ $\quad [\check{\mathbf{h}} \times \boldsymbol{n} = 0$ on $\partial\Omega$, $\check{\mathbf{h}} \in \boldsymbol{H}_0(\mathbf{curl},\Omega)$ in terms of vector proxies $]$.

Magnetostatic BVP:

$$\mathrm{d}_1\mathbf{h} = \mathbf{j}\,, \quad \mathrm{d}_2\mathbf{b} = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^3\,, \quad \mathbf{h}|_{\partial\Omega} = 0 \quad \text{on} \quad \partial\Omega\,, \tag{13.3.3.9}$$

[PMC]

weak MLs:

$$\int_\Omega \mathbf{b} \wedge \mathbf{h}' = \mathrm{m}_\mu(\mathbf{h},\mathbf{h}') \quad \forall \mathbf{h}' \in L^2\Lambda^1(\Omega) \quad , \quad \int_\Omega \mathbf{h} \wedge \mathbf{b}' = \mathrm{m}^*_\mu(\mathbf{b},\mathbf{b}') \quad \forall \mathbf{b}' \in L^2\Lambda^2(\Omega)\,.$$

13.3.3.1 MS: Formulations Based on Scalar Potential

Assumption 13.3.3.7. Simple topology of Ω

We assume that the assumptions of Thm. 13.1.4.81 on Ω for $\ell = 1,2$ are satisfied:
- Every oriented closed 2-surface Ω is the boundary of a sub-domain $\subset \Omega$.
- Every closed directed curve in Ω is the boundary of an oriented surface $\subset \Omega$.

$$\left.\begin{aligned} d_2 j &= 0 \\ j|_{\partial\Omega} &= 0 \end{aligned}\right\} \overset{P.T.}{\Longrightarrow} \exists h_s \in H\Lambda^1_0(d,\Omega) : d_1 h_s = j$$

↳ current vector potential

h solves (9): $\quad d_1(h - h_s) = 0\,, \quad h - h_s|_{\partial\Omega} = 0$

$$\overset{P.T.}{\Longrightarrow} \quad \exists v \in H\Lambda^0_0(d,\Omega) : d_0 v = h - h_s$$

↳ magnetic scalar potential

$$\mathbf{h} = \mathbf{h}_s - \mathrm{d}_0 v \quad \text{and} \quad \mathrm{d}_e\mathbf{b} = 0 \quad \text{and} \quad \int_\omega \mathbf{b} \wedge \mathbf{h}' = \mathrm{m}_\mu(\mathbf{h},\mathbf{h}') \quad \forall \mathbf{h}' \in L^2\Lambda^1(\Omega)\,.$$

$$\mathbf{h} = \mathbf{h}_s - \mathrm{d}_0 v \quad \text{and} \quad \mathrm{d}_e \mathbf{b} = 0 \quad \text{and} \quad \int_\omega \mathbf{b} \wedge \mathbf{h}' = \mathrm{m}_\mu(\mathbf{h}, \mathbf{h}') \quad \forall \mathbf{h}' \in L^2\Lambda^1(\Omega) .$$

$h' := d_0 v'$

$v' \in H_0\Lambda^0(d, \Omega)$

$$\mathrm{m}_\mu(\mathbf{h}_s - \mathrm{d}_0 v, \mathrm{d}_0 v') = \int_\omega \mathbf{b} \wedge \mathrm{d}_0 v' = \int_\omega \underbrace{\mathrm{d}_2 \mathbf{b}}_{=0} \wedge v' = 0 \quad \forall v' \in H_0\Lambda^0(\mathrm{d}, \Omega) .$$

i.b.p. [no boundary terms, since $v'_{|\partial\Omega} = 0$]

Seek $v \in H_0\Lambda^0(\mathrm{d}, \Omega)$ such that

$$\mathrm{m}_\mu(\mathrm{d}_0 v, \mathrm{d}_0 v') = \mathrm{m}_\mu(\mathbf{h}_s, \mathrm{d}_0 v') \quad \forall v' \in H_0\Lambda^0(\mathrm{d}, \Omega) . \tag{13.3.3.12}$$

For local, linear ML, v.p.

$\breve{v} \in H_0^1(\Omega)$:

$$\int_\Omega (\boldsymbol{\mu}(\boldsymbol{x}) \,\mathbf{grad}\, \breve{v}(\boldsymbol{x})) \cdot \mathbf{grad}\, \breve{v}'(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega (\boldsymbol{\mu}(\boldsymbol{x}) \mathbf{h}_s(\boldsymbol{x})) \cdot \mathbf{grad}\, \breve{v}'(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \quad \forall \breve{v}' \in H_0^1(\Omega) . \tag{13.3.3.13}$$

→ 2nd-order elliptic BVP in weak form

§ 13.3.3.14 FE Galerkin discretization

$H_0^1(\Omega) \longleftarrow W_0^0(\mathcal{M}) = S_{1,0}^0(\mathcal{M})$

→ Chapter 3 : Lagrangian FE

Rem 13.3.3.16: How to find $h_S \in H_0\Lambda^1(d,\Omega)$

$\longrightarrow$ Idea: seek $h_S \in \mathcal{W}_0^1(\mathcal{M})$, after interpolating j into $\mathcal{W}_0^2(\mathcal{M})$ by sampling face fluxes

Then solve $\quad \mathbf{D}_1 \vec{\mathbf{h}}_S = \vec{\mathbf{j}} = \underset{\uparrow \text{ face flux sampling op.}}{\mathsf{S}_2} j$

[Note: $\mathbf{D}_2 \vec{\mathbf{j}} = 0$]

$\longrightarrow$ Graph-theoretic spanning-tree algorithms

Effort $= O(N_1)$ for $N_1 \to \infty$ & families of uniformly shape regular meshes.