ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 13.3.3.2: Magnetostatics: Vector-Potential-Based Variational Formulations

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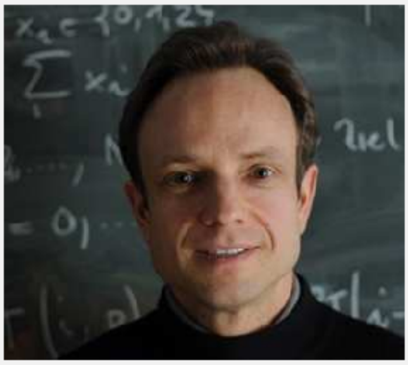

Dependencies. [Lecture → Section 1.8], [Lecture → Section 13.3.1.2], [Lecture → Section 13.1.4], [Lecture → Section 13.1.5].

Duration: 40 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XIII Finite-Element Exterior Calculus (FEEC)

13.3. Whitney FEM for Computational Electromagnetism

13.3.3. Magnetostatics (MS)

13.3.3.2. MS: Vector-Potential-Based Formulation

BVP:

$$\mathrm{d}_1\mathbf{h} = \mathbf{j}\,,\quad \mathrm{d}_2\mathbf{b} = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^3\,,\quad \mathbf{h}|_{\partial\Omega} = 0 \quad \text{on} \quad \partial\Omega\,, \tag{13.3.3.9}$$

$$\int_\Omega \mathbf{b} \wedge \mathbf{h}' = \mathrm{m}_\mu(\mathbf{h},\mathbf{h}') \quad \forall \mathbf{h}' \in L^2\Lambda^1(\Omega)\quad,\quad \underline{\int_\Omega \mathbf{h} \wedge \mathbf{b}' = \mathrm{m}^*_\mu(\mathbf{b},\mathbf{b}')} \quad \forall \mathbf{b}' \in L^2\Lambda^2(\Omega)\,.$$

$b = d_1 a$, $a \mathrel{\hat{=}}$ magnetic vector potential

$$\underbrace{\mathrm{d}_1\mathbf{h} = \mathbf{j}}_{(AL)}\,,\quad \mathbf{b} = \mathrm{d}_1\mathbf{a} \quad \text{in} \quad \Omega \subset \mathbb{R}^3\,,\quad \mathbf{h}|_{\partial\Omega} = 0 \quad \text{on} \quad \partial\Omega\,. \tag{13.3.3.19}$$

$$\int_\Omega j \wedge a' \overset{AL}{=} \int_\Omega d_1 h \wedge a' \overset{i.b.p.}{=} \int_\Omega h \wedge d_1 a' + \overbrace{\cancel{\int_{\partial\Omega} h \wedge a'}}^{=0} \quad [h|_{\partial\Omega} = 0]$$

$$\overset{ML}{=} m^*_\mu(b, d a') \overset{pot.}{=} m^*_\mu(d_1 a, d a') \quad \forall a'$$

$$\boxed{\mathbf{a} \in H\Lambda^1(\mathrm{d},\Omega): \quad \mathrm{m}^*_\mu(\mathrm{d}_1\mathbf{a}, \mathrm{d}_1\mathbf{a}') = \int_\Omega \mathbf{j} \wedge \mathbf{a}' \quad \forall \mathbf{a}' \in H\Lambda^1(\mathrm{d},\Omega)} \,. \tag{13.3.3.21}$$

LLML & v.p.:

$$\breve{\mathbf{a}} \in \boldsymbol{H}(\mathbf{curl},\Omega): \quad \int_\Omega \mu(\boldsymbol{x})^{-1}\, \mathbf{curl}\, \breve{\mathbf{a}}(\boldsymbol{x}) \cdot \mathbf{curl}\, \breve{\mathbf{a}}'(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{j}(\boldsymbol{x}) \cdot \breve{\mathbf{a}}'(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \forall \breve{\mathbf{a}}' \in \boldsymbol{H}(\mathbf{curl},\Omega)\,. \tag{13.3.3.22}$$

PMC b.c.? ≙ natural b.c. ("hidden in (21)")

§ 13.3.3.23 Gauge Freedom

If a solves (21) $\Rightarrow$ $a + d_0 v$ solves (21) $\forall v \in H\Lambda^0(d,\Omega)$

$$\mathcal{N}(d_1) = d_0 H\Lambda^0(d,\Omega)$$

Assumption 13.3.3.7. Simple topology of Ω

We assume that the assumptions of Thm. 13.1.4.81 on Ω for $\ell = 1, 2$ are satisfied:

- Every oriented closed 2-surface Ω is the boundary of a sub-domain $\subset \Omega$.
- Every closed directed curve in Ω is the boundary of an oriented surface $\subset \Omega$.

(21): Non-uniqueness

Existence

$$\text{r.h.s}(d_0 v') = \int_\Omega g \wedge d_0 v'$$

$$\overset{\text{i.p.b.}}{=} -\int_\Omega \underbrace{d_2 g}_{=0} \wedge v' + \int_{\partial\Omega} \underbrace{g \wedge v}_{g|_{\partial\Omega}=0} = 0$$

$\rightarrow$ Consistent r.h.s $\Rightarrow$ Existence of sols

<u>Restoring uniqueness = "suppressing kernel"</u>

Simple setting (Ex. 13.3.3.25)

$$Z^T\vec{\varphi} = 0$$

$$\Updownarrow$$

$$\mathbf{A}\vec{\xi} = \vec{\varphi}\,, \quad \mathbf{A} = \mathbf{A}^\top \in \mathbb{R}^{N,N}\,, \quad \vec{\varphi} \in \mathbb{R}^N\,, \quad \mathcal{N}(\mathbf{A}) \neq \{\mathbf{0}\}\,, \quad \vec{\varphi} \perp \mathcal{N}(\mathbf{A})\,.$$

$$Z \in \mathbb{R}^{N,M}: \ \mathcal{R}(A) = \mathcal{N}(A)\,, \ M = \dim \mathcal{N}(A)$$

Restore uniqueness of solutions by enforcing $\underbrace{\text{orthogonality to } \mathcal{N}(\mathbf{A})}_{Z^T\vec{\xi} = 0}$!

$$A\vec{\xi} + Z\vec{\pi} = \vec{\varphi}$$

$$Z^T\vec{\xi} = 0$$

$$\cancel{Z^T A}\vec{\xi} + \underbrace{Z^T Z}_{\text{invertible}}\vec{\pi} = \cancel{Z^T\vec{\varphi}} = 0 \Rightarrow \vec{\pi} = 0$$

Augmented LSE:

$$\begin{bmatrix} \mathbf{A} & \mathbf{Z} \\ \mathbf{Z}^\top & \mathbf{O} \end{bmatrix} \begin{bmatrix} \vec{\xi} \\ \vec{\pi} \end{bmatrix} = \begin{bmatrix} \vec{\varphi} \\ \mathbf{0} \end{bmatrix}. \qquad (13.3.3.26)$$

regular square matrix

Apply this idea to

$$\mathbf{a} \in H\Lambda^1(\mathrm{d},\Omega): \quad \mathrm{m}^*_\mu(\mathrm{d}_1\mathbf{a}, \mathrm{d}_1\mathbf{a}') = \int_\Omega \mathbf{j} \wedge \mathbf{a}' \quad \forall \mathbf{a}' \in H\Lambda^1(\mathrm{d},\Omega), \qquad (13.3.3.21)$$

Orthogonality requires inner product $m(\cdot,\cdot)$ on $H\Lambda^1(d,\Omega)$

Use any L^2-type i.p.!

Important: "Kernel representation"

$$\mathcal{N}(d_1) = d_0 H\Lambda^0(d,\Omega)$$

Seek $\mathbf{a} \in H\Lambda^1(\mathrm{d},\Omega)$, $p \in H_*\Lambda^0(\mathrm{d},\Omega) := \{q \in H\Lambda^0(\mathrm{d},\Omega) : \int_\Omega \mathrm{v.p.}(q)(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0\}$

$$\begin{aligned} \mathrm{m}^*_\mu(\mathrm{d}_1\mathbf{a}, \mathrm{d}_1\mathbf{a}') + \mathrm{m}(\mathbf{a}', \mathrm{d}_0 p) &= \int_\Omega \mathbf{j}\wedge\mathbf{a}' \quad \forall \mathbf{a}' \in H\Lambda^1(\mathrm{d},\Omega), \\ \mathrm{m}(\mathbf{a}, \mathrm{d}_0 p') &= 0 \quad \forall p' \in H_*\Lambda^0(\mathrm{d},\Omega). \end{aligned} \qquad (13.3.3.28)$$

enforces $m(\cdot,\cdot)$-orthogonality to $\mathcal{N}(d_1)$

$p = 0$!

Note: $\mathcal{N}(d_0) = \mathcal{P}_0\Lambda^0(\Omega)$

▷ Enforce uniqueness of p by orthogonality to constants!

LLML, v.p. notation:

Seek $\breve{\mathbf{a}} \in \boldsymbol{H}(\mathbf{curl},\Omega)$, $\breve{p} \in H^1_*(\Omega) := \{v \in H^1(\Omega) : \int_\Omega v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0\}$ such that

$$\begin{aligned}
\int_\Omega \mu(\boldsymbol{x})^{-1}\,\mathbf{curl}\,\breve{\mathbf{a}}(\boldsymbol{x})\cdot\mathbf{curl}\,\breve{\mathbf{a}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \;+\; \int_\Omega \breve{\mathbf{a}}'(\boldsymbol{x})\cdot\mathbf{grad}\,\breve{p}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \breve{\mathbf{j}}(\boldsymbol{x})\cdot\breve{\mathbf{a}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,,\\
\int_\Omega \breve{\mathbf{a}}(\boldsymbol{x})\cdot\mathbf{grad}\,\breve{p}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= 0\,.
\end{aligned} \tag{13.3.3.29}$$

for all $\breve{\mathbf{a}}' \in \boldsymbol{H}(\mathbf{curl},\Omega)$, $\breve{p}' \in H^1_*(\Omega)$.

§ 13.3.3.30 Galerkin FEM

▽ Discrete kernel representation

$$\{v_h \in \mathcal{W}^1(\mathcal{M}) : d_1 v_h = 0\} = d_0 \mathcal{W}^0(\mathcal{M})$$

Step I: f.d. subspaces → Whitney FE spaces

Seek $\mathbf{a}_h \in \mathcal{W}^1(\mathcal{M})$, $p_h \in \mathcal{W}^0(\mathcal{M}) \cap H_*\Lambda^0(\mathsf{d},\Omega)$ such that

$$\begin{aligned}
\mathsf{m}^*_\mu(\mathsf{d}_1\mathbf{a}_h, \mathsf{d}_1\mathbf{a}'_h) \;+\; \mathsf{m}(\mathbf{a}'_h, \mathsf{d}_0 p_h) &= \int_\Omega \mathbf{j}\wedge\mathbf{a}'_h \quad \forall \mathbf{a}'_h \in \mathcal{W}^1(\mathcal{M})\,,\\
\underbrace{\mathsf{m}(\mathbf{a}_h, \mathsf{d}_0 p'_h) = 0}_{\hat{=}\ \text{orthogonality to (discrete) kernel}} &\quad\;\; \forall p'_h \in \mathcal{W}^0(\mathcal{M}) \cap H_*\Lambda^0(\mathsf{d},\Omega)\,.
\end{aligned} \tag{13.3.3.32}$$

≙ orthogonality to (discrete) kernel

Vanishing mean constraint:

Seek $\mathbf{a}_h \in \mathcal{W}^1(\mathcal{M})$, $p_h \in \mathcal{W}^0(\mathcal{M})$, and $\lambda \in \mathbb{R}$ such that

$$\begin{aligned}
\mathsf{m}^*_\mu(\mathsf{d}_1\mathbf{a}_h, \mathsf{d}_1\mathbf{a}'_h) + \mathsf{m}(\mathbf{a}'_h, \mathsf{d}_0 p_h) &= \int_\Omega \mathbf{j} \wedge \mathbf{a}'_h\,, \\
\mathsf{m}(\mathbf{a}_h, \mathsf{d}_0 p'_h) \qquad\qquad + \lambda \int_\Omega p_h(\boldsymbol{x})'\,\mathrm{d}\boldsymbol{x} &= 0\,, \\
\int_\Omega p_h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= 0\,.
\end{aligned} \tag{13.3.3.33}$$

for all $\check{\mathbf{a}}'_h \in \mathcal{W}^1(\mathcal{M})$, $p'_h \in \mathcal{W}^0(\mathcal{M})$.

$\lambda = 0$

"Orthogonality to const."

Step II: Ordered bases {Whitney 1/0-forms}

$\triangleright$ LSE:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^\top & \mathbf{0} \\ \mathbf{B} & \mathbf{O} & \mathbf{c} \\ \mathbf{0} & \mathbf{c}^\top & 0 \end{bmatrix} \begin{bmatrix} \vec{\mathbf{a}}_h \\ \vec{p}_h \\ \lambda \end{bmatrix} = \begin{bmatrix} \vec{\boldsymbol{\psi}} \\ \mathbf{0} \\ 0 \end{bmatrix}, \tag{13.3.3.34}$$

$\hookrightarrow$ dummy unknown $= 0$