ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 13.3.3.3–Section 13.3.3.4: Saddle-Point Variational Problems, Discrete LBB-Conditions for $\mathbf{a}$-Based Magnetostatics

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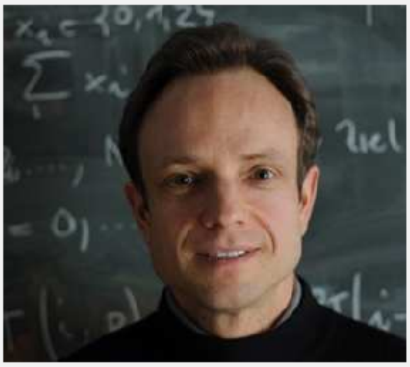

Dependencies. [Lecture → § 12.2.2.20], [Lecture → § 12.3.3.1], [Lecture → Section 13.3.3.2]

Duration: 60 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

XIII Finite-Element Exterior Calculus (FEEC)

13.3. Whitney FEM for Computational Electromagnetism

13.3.3. Magnetostatics

MS a-based BVP, variational formulation:

Seek $\mathbf{a} \in H\Lambda^1(\mathrm{d},\Omega)$, $p \in H_*\Lambda^0(\mathrm{d},\Omega) := \{q \in H\Lambda^0(\mathrm{d},\Omega) : \int_\Omega \mathrm{v.p.}(q)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0\}$

$$\begin{aligned} \mathsf{m}^*_\mu(\mathrm{d}_1\mathbf{a},\mathrm{d}_1\mathbf{a}') + \mathsf{m}(\mathbf{a}',\mathrm{d}_0 p) &= \int_\Omega \mathbf{j}\wedge\mathbf{a}' \quad \forall \mathbf{a}' \in H\Lambda^1(\mathrm{d},\Omega)\,, \\ \mathsf{m}(\mathbf{a},\mathrm{d}_0 p') &= 0 \quad \forall p' \in H_*\Lambda^0(\mathrm{d},\Omega)\,. \end{aligned} \tag{13.3.3.28}$$

13.3.3.3. Saddle Point Variational Problems

$$\text{Seek } \begin{matrix} v \in U \\ p \in Q \end{matrix} : \quad \begin{aligned} \mathsf{a}(v,w) + \mathsf{b}(w,p) &= \ell(w) \quad \forall w \in U\,, \\ \mathsf{b}(v,q) &= g(q) \quad \forall q \in Q\,, \end{aligned} \tag{12.2.2.49}$$

where
- ✦ U and Q are Hilbert spaces, with norms $\|\cdot\|_U$ and $\|\cdot\|_Q$, respectively,
- ✦ $\mathsf{a} : U \times U \to \mathbb{R}$ and $\mathsf{b} : U \times Q \to \mathbb{R}$ are continuous/bounded bilinear forms,

$$\begin{aligned} \exists C_a > 0: &\quad |\mathsf{a}(v,w)| \le C_a \|v\|_U \|w\|_U \quad \forall v,w \in U \quad \text{and} \\ \exists C_b > 0: &\quad |\mathsf{b}(v,q)| \le C_b \|v\|_U \|q\|_Q \quad \forall v \in U,\ q \in Q\,, \end{aligned} \tag{13.3.3.36}$$

- ✦ $\ell : U \to \mathbb{R}$ and $g : Q \to \mathbb{R}$ are continuous/bounded linear forms.

$$\mathcal{N}(b) := \{ u \in \mathcal{U} : b(u,q) = 0 \ \forall q \in Q \} \subset \mathcal{U}$$

Theorem 12.2.2.23. Ladyzhenskaya–Babuška–Brezzi conditions (LBB-conditions)

If the following two conditions are satisfied,

(i) *the bilinear form* a *is* $\mathcal{N}(\mathsf{b})$*-elliptic* (***ellipticity on the kernel***)

$$\exists \alpha > 0: \quad |\mathsf{a}(v,v)| \geq \alpha \|v\|_U^2 \quad \forall v \in \mathcal{N}(\mathsf{b}), \tag{LBB1}$$

(ii) *the bilinear form* b *satisfies the* ***inf-sup condtion***

$$\exists \beta > 0: \quad \sup_{v \in U} \frac{|\mathsf{b}(v,q)|}{\|v\|_U} \geq \beta \|q\|_Q \quad \forall q \in Q, \tag{LBB2}$$

then the variational saddle point problem (12.2.2.49) *possesses a unique solution* $(v,p) \in U \times Q$*, which satisfies*

Stability:

$$\|v\|_U + \|p\|_Q \leq C \left(\sup_{w \in U} \frac{|\ell(w)|}{\|w\|_U} + \sup_{q \in Q} \frac{|g(q)|}{\|q\|_Q} \right), \quad \to [12.2.2.24]$$

with $C > 0$ *depending only on* C_a, C_b *and* α, β.

Explanation in Euclidean setting

$$\begin{aligned} \vec{v} \in \mathbb{R}^m = \mathcal{U} : \quad & \mathsf{a}(\vec{v},\vec{\eta}) + \mathsf{b}(\vec{\eta},\vec{\pi}) = \ell(\vec{\eta}) \quad \forall \vec{\eta} \in \mathbb{R}^m, \\ \vec{\pi} \in \mathbb{R}^n = Q : \quad & \mathsf{b}(\vec{v},\vec{\omega}) = g(\vec{\omega}) \quad \forall \vec{\omega} \in \mathbb{R}^n. \end{aligned} \tag{12.2.2.30}$$

$$\exists \mathbf{A} \in \mathbb{R}^{m,m}: \quad \mathsf{a}(\vec{v},\vec{\eta}) = \vec{\eta}^\top \mathbf{A} \vec{v} \quad \forall \vec{\eta}, \vec{v} \in \mathbb{R}^m,$$

$$\exists \mathbf{B} \in \mathbb{R}^{n,m}: \quad \mathsf{b}(\vec{v},\vec{\omega}) = \vec{\omega}^\top \mathbf{B} \vec{v} \quad \forall \vec{v} \in \mathbb{R}^m, \ \vec{\omega} \in \mathbb{R}^n,$$

$$\exists \vec{\varphi} \in \mathbb{R}^m: \quad \ell(\vec{\eta}) = \vec{\eta}^\top \vec{\varphi} \quad \forall \vec{\eta} \in \mathbb{R}^m \quad , \quad \exists \vec{\gamma} \in \mathbb{R}^m: \quad g(\vec{\omega}) = \vec{\omega}^\top \vec{\gamma} \quad \forall \vec{\omega} \in \mathbb{R}^n,$$

$$\begin{aligned} \vec{v} \in \mathbb{R}^m \\ \vec{\pi} \in \mathbb{R}^n \end{aligned} : \quad \begin{aligned} \vec{\eta}^\top \mathbf{A} \vec{v} + \vec{\eta}^\top \mathbf{B}^\top \vec{\pi} &= \vec{\eta}^\top \vec{\varphi} \quad \forall \vec{\eta} \in \mathbb{R}^m, \\ \vec{\omega}^\top \mathbf{B} \vec{v} &= \vec{\omega}^\top \vec{\gamma} \quad \forall \vec{\omega} \in \mathbb{R}^n. \end{aligned} \tag{12.2.2.31}$$

$$\begin{array}{l}\vec{v}\in\mathbb{R}^m\\ \vec{\pi}\in\mathbb{R}^n\end{array}:\quad \begin{array}{rcl}\mathbf{A}\vec{v}+\mathbf{B}^\top\vec{\pi} &=& \vec{\varphi}\;,\\ \mathbf{B}\vec{v} &=& \vec{\gamma}\;,\end{array} \tag{12.2.2.32}$$

$\hat{=}$ Saddle point LSE

$m > n$:

$$\underbrace{\begin{bmatrix}\mathbf{A} & \mathbf{B}^\top\\ \mathbf{B} & \mathbf{O}\end{bmatrix}}_{m,\ n}\begin{bmatrix}\vec{v}\\ \vec{\pi}\end{bmatrix}=\begin{bmatrix}\vec{\varphi}\\ \vec{\gamma}\end{bmatrix}. \tag{12.2.2.33}$$

$\mathcal{N}(b) = \{\vec{\eta}\in\mathbb{R}^m :\ \vec{\omega}^T B\vec{\eta} = 0\quad \forall\vec{\omega}\in\mathbb{R}^n\}$

$= \mathcal{N}(B)$

$= \mathcal{R}(Z)$, $Z\in\mathbb{R}^{m,k}$, orthonormal columns
$k = \dim\mathcal{N}(B)$

(LBB2)

$$\exists\beta>0:\quad \sup_{\vec{\eta}\in\mathbb{R}^m}\frac{\vec{\eta}^\top\mathbf{B}^\top\vec{\omega}}{\|\vec{\eta}\|_2}\geq\beta\|\vec{\omega}\|_2\quad\forall\vec{\omega}\in\mathbb{R}^n\,. \tag{12.2.2.35}$$

$\Rightarrow\ \mathcal{N}(B^T) = \{0\} \Rightarrow \mathrm{Rank}(B^T) = n \Rightarrow k = m-n$

LBB2 $\Rightarrow$ $m \geq n$

$\mathcal{R}(B) \overset{\uparrow}{=} \mathcal{N}(B^T)^\perp = \{0\}^\perp = \mathbb{R}^n$

Thm. from LA

$\triangleright$ $\exists \vec{v}_* \in \mathbb{R}^m$: $B\vec{v}_* = \vec{g} \in \mathbb{R}^n$

Set $\vec{v} := \vec{v}_* + Z\vec{\varphi}$, $\vec{\varphi} \in \mathbb{R}^k$

Into 1st equ. :

$Z^T \cdot \mid A(\vec{v}_* + Z\vec{\varphi}) + B^T\vec{\pi} = \vec{\varphi}$

$(*)$ $Z^TAZ\vec{\varphi} + \underbrace{Z^TB^T}_{=0}\vec{\pi} = Z^T\vec{\varphi} - Z^TA\vec{v}_*$

LBBI: $|\vec{p}^T \underbrace{Z^TAZ}_{\text{s.p.d.}} \vec{p}| \geq \alpha \|Z\vec{p}\|^2 = \alpha \|\vec{p}\|^2$

$\exists_1 \vec{\varphi}_*$ solving $(*)$: $\vec{v} := \vec{v}_* + Z\vec{\varphi}_*$

$\exists \vec{\pi}$? $B^T\vec{\pi} = \vec{\varphi} - A\vec{v} \overset{?}{\in} \mathcal{R}(B^T)$

$\hookrightarrow$ unique, since $\mathcal{N}(B^T) = \{0\}$

$\mathcal{R}(B^T) = \mathcal{N}(B)^\perp$:

$Z^T(\vec{\varphi} - A\vec{v}) = Z^T\vec{\varphi} - Z^TAZ\vec{\varphi}_* - Z^TA\vec{v}_* \overset{(*)}{=} 0$

$\triangleright$ $\exists \vec{\pi}$

Galerkin discretization:

$\mathcal{U}_h \subset \mathcal{U}$, $\dim \mathcal{U}_h = m$

$Q_h \subset Q$, $\dim Q_h = n$

LBB2 in discrete setting $\Rightarrow$ $m \geq n$

Discrete LBB conditions $\Rightarrow$ quasi-optimality

Theorem 13.3.3.40. Galerkin discretization error estimate for discrete variational saddle point problems

Let the assumptions of Thm. 12.2.2.23 be satsified. Also assume discrete counterparts of (LBB1) *and* (LBB2), *namely the* ***discrete LBB-conditions***

$$\exists \alpha_h > 0: \qquad |\mathsf{a}(v_h, v_h)| \geq \alpha_h \|v_h\|_U^2 \quad \forall v_h \in \underline{\mathcal{N}_h(\mathsf{b})}\,, \tag{LBB1$_h$}$$

$$\exists \beta_h > 0: \qquad \sup_{v_h \in U_h} \frac{|\mathsf{b}(v_h, q_h)|}{\|v\|_U} \geq \beta_h \|q_h\|_Q \quad \forall q_h \in Q_h\,. \tag{LBB2$_h$}$$

then the discrete variational saddle point problem (12.2.2.49) *possesses a unique solution* $(v_h, p_h) \in U_h \times Q_h$, *which satisfies the Galerkin discretization error estimate*

$$\|v - v_h\|_U + \|p - p_h\|_Q \leq C \left(\inf_{u_h \in U_h} \|v - u_h\|_U + \inf_{q_h \in Q_h} \|p - q_h\|_Q \right), \tag{13.3.3.41}$$

where $(v, p) \in U \times Q$ *solves* (12.2.2.49) *and with a constant* $C > 0$ *depending only on* C_a, C_b *and* α_h, β_h.

‖ discretization error ‖ ‖ best approximation error ‖

$\mathcal{N}_h(b) = \{ u_h \in \mathcal{U}_h : b(u_h, q_h) = 0 \ \forall q_h \in Q_h \} \subset \mathcal{U}_h$

Note: Not necessarily $\mathcal{N}_h(b) \subset \mathcal{N}(b)$

Step II of G.D.:

ordered bases $\mathfrak{B}_U := \{b_h^1, \dots, b_h^N\}$, $\mathfrak{B}_Q := \{\beta_h^1, \dots, \beta_h^M\}$ of U_h, $N := \dim U_h$, Q_h, $M := \dim Q_h$,

saddle point LSE:
$$\boxed{\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{\pi} \end{bmatrix} = \begin{bmatrix} \vec{\psi} \\ \vec{\varphi} \end{bmatrix}}, \qquad (12.3.0.7)$$

e.g. $\mathbf{B} = \left[b(b_h^j, \beta_h^i) \right]_{\substack{j=1,\dots,N \\ i=1,\dots,M}}$

1.3.3.3.4 Discrete LBB-Conditions for a-Based Magnetostatics

Discrete MS var. SPP:

$$\begin{aligned} \mathsf{m}_\mu^*(\mathsf{d}_1\mathbf{a}_h, \mathsf{d}_1\mathbf{a}_h') \;+\; \mathsf{m}(\mathbf{a}_h', \mathsf{d}_0 p_h) &= \int_\Omega \mathbf{j} \wedge \mathbf{a}_h' \quad \forall \mathbf{a}_h' \in \mathcal{W}^1(\mathcal{M}), \\ \mathsf{m}(\mathbf{a}_h, \mathsf{d}_0 p_h') &= 0 \quad \forall p_h' \in \mathcal{W}^0(\mathcal{M}) \cap H_*\Lambda^0(\mathsf{d},\Omega). \end{aligned} \qquad (13.3.3.32)$$

- for the spaces: $U \leftrightarrow H\Lambda^1(\mathsf{d},\Omega)$ and $Q \leftrightarrow H_*\Lambda^0(\mathsf{d},\Omega)$,
- for the (bi-)linear forms

$$\begin{aligned} \mathsf{a} &\leftrightarrow (\mathbf{a},\mathbf{a}') \in H\Lambda^1(\mathsf{d},\Omega) \times H\Lambda^1(\mathsf{d},\Omega) \mapsto \mathsf{m}_\mu^*(\mathsf{d}_1\mathbf{a}, \mathsf{d}_1\mathbf{a}'), \\ \mathsf{b} &\leftrightarrow (\mathbf{a},p') \in H\Lambda^1(\mathsf{d},\Omega) \times H_*\Lambda^0(\mathsf{d},\Omega) \mapsto \mathsf{m}(\mathbf{a}, \mathsf{d}_0 p'), \\ \ell &\leftrightarrow \mathbf{a}' \in H\Lambda^1(\mathsf{d},\Omega) \mapsto \int_\Omega \mathbf{j} \wedge \mathbf{a}', \end{aligned}$$

- and for the Galerkin trial/test spaces: $U_h \leftrightarrow \mathcal{W}^1(\mathcal{M})$, $Q_h \leftrightarrow \mathcal{W}^0(\mathcal{M}) \cap H_*\Lambda^0(\mathsf{d},\Omega)$, spaces of Whitney finite elements.

(Discrete) LBB-conditions:

LBB2 / LBB2$_h$:

$$\Leftrightarrow \quad \exists \beta > 0: \quad \sup_{\mathbf{a}' \in H\Lambda^1(\mathrm{d},\Omega)} \frac{|\mathsf{m}(\mathbf{a}', \mathrm{d}_0 p)|}{\|\mathbf{a}'\|_{H\Lambda^1(\mathrm{d},\Omega)}} \geq \beta \|p\|_{H\Lambda^0(\mathrm{d},\Omega)} \quad \forall p \in H_*\Lambda^0(\mathrm{d},\Omega)\,. \tag{13.3.3.53}$$

"Candidate": $a' := d_0 p \in H\Lambda^1(d,\Omega)$

$$\sup_{\mathbf{a}' \in H\Lambda^1(\mathrm{d},\Omega)} \frac{|\mathsf{m}(\mathbf{a}', \mathrm{d}_0 p)|}{\|\mathbf{a}'\|_{H\Lambda^1(\mathrm{d},\Omega)}} \geq \frac{\mathsf{m}(\mathrm{d}_0 p, \mathrm{d}_0 p)}{\|\mathrm{d}_0 p\|_{H\Lambda^1(\mathrm{d},\Omega)}} \geq \|\mathrm{d}_0 p\|_{L^2\Lambda^1(\Omega)} \geq C\|p\|_{H\Lambda^0(\mathrm{d},\Omega)}\,,$$

[$m(\cdot,\cdot) \leftrightarrow L^2\Lambda^1(\Omega)$ - inner product]

by vanishing mean
Thm 1.8.0.20

Discrete:

$$\Leftrightarrow \quad \exists \beta_h > 0: \quad \sup_{\mathbf{a}'_h \in \mathcal{W}^1(\mathcal{M})} \frac{|\mathsf{m}(\mathbf{a}'_h, \mathrm{d}_0 p_h)|}{\|\mathbf{a}'_h\|_{H\Lambda^1(\mathrm{d},\Omega)}} \geq \beta_h \|p_h\|_{H\Lambda^0(\mathrm{d},\Omega)} \quad \forall p_h \in \mathcal{W}^0(\mathcal{M}) \cap H_*\Lambda^0(\mathrm{d},\Omega)\,. \tag{13.3.3.54}$$

Use $d_0 W^0(\mathcal{M}) \subset W^1(\mathcal{M})$: $a_h' := d_0 p_h$

$$\sup_{\mathbf{a}'_h \in \mathcal{W}^1(\mathcal{M})} \frac{|\mathsf{m}(\mathbf{a}'_h, \mathrm{d}_0 p_h)|}{\|\mathbf{a}'_h\|_{H\Lambda^1(\mathrm{d},\Omega)}} \geq \frac{|\mathsf{m}(\mathrm{d}_0 p_h, \mathrm{d}_0 p_h)|}{\|\mathrm{d}_0 p'_h\|_{H\Lambda^1(\mathrm{d},\Omega)}} \geq \|\mathrm{d}_0 p_h\|_{L^2\Lambda^1(\Omega)} \geq C\|p_h\|_{H\Lambda^0(\mathrm{d},\Omega)}\,.$$

LBBI / LBBI_h :

Assumption 13.3.3.7. Simple topology of Ω

We assume that the assumptions of Thm. 13.1.4.81 on Ω for $\ell = 1, 2$ are satisfied:

- Every oriented closed 2-surface Ω is the boundary of a sub-domain $\subset \Omega$.
- Every closed directed curve in Ω is the boundary of an oriented surface $\subset \Omega$.

$$\mathcal{N}(\mathsf{d}_1) := \left\{ \mathbf{v} \in H\Lambda^1(\mathsf{d}, \Omega) : \mathsf{d}_1 \mathbf{v} = 0 \right\} = \mathsf{d}_0 H\Lambda^0(\mathsf{d}, \Omega) . \tag{13.3.3.57}$$

$$\mathcal{N}(\mathsf{b}) = \left\{ \mathbf{v} \in H\Lambda^1(\mathsf{d}, \Omega) : \mathsf{m}(\mathbf{v}, \mathsf{d}_0 q) = 0 \ \forall q \in H\Lambda^0(\mathsf{d}, \Omega) \right\} . \tag{13.3.3.56}$$

$$= \{ \underline{v} \in H\Lambda^1(d, \Omega) : m(v, z) = 0 \ \forall z \in \mathcal{N}(d_1) \}$$

$\rightarrow$ m-orth. complement of $\mathcal{N}(d_1)$

[" $\Rightarrow$ d_1 stably invertible on $\mathcal{N}(b)$ "]

Theorem 13.3.3.58. Poincaré-Friedrichs-type inequality for d_1

Under Ass. 13.3.3.7 and with a constant $C_1 > 0$ depending only on Ω (and on the definition of the L^2-norm) holds true

$$\|\mathbf{v}\|_{L^2\Lambda^1(\Omega)} \leq C_1 \|\mathsf{d}_1 \mathbf{v}\|_{L^2\Lambda^2(\Omega)} \quad \forall \mathbf{v} \in \mathcal{N}(\mathsf{b}) . \tag{13.3.3.59}$$

$\Rightarrow$ LBBI :

Theorem 13.3.3.64. Discrete Poincaré-Friedrichs-type inequality for d_1

Taking for granted Ass. 13.3.3.7 we have

$$\mathsf{m}^*_\mu(\mathsf{d}_1 \mathbf{a}'_h, \mathsf{d}_1 \mathbf{a}'_h) \geq \alpha \|\mathbf{a}'_h\|^2_{L^2\Lambda^1(\Omega)} \quad \forall \mathbf{a}'_h \in \mathcal{N}_h(\mathsf{b}) , \tag{13.3.3.65}$$

with a constant $\alpha > 0$ depending only on Ω, $\mu^\pm_*$, and the shape regularity measure of $\mathcal{M}$ (and on the definition of the L^2-norm).

Note: Not a consequence of Thm. 13.3.3.58, because $\mathcal{N}_h(b) \not\subset \mathcal{N}(b)$

▷ Quasi-optimality

$$\|\mathbf{a}-\mathbf{a}_h\|_{H\Lambda^1(\mathrm{d},\Omega)} + \|p-p_h\|_{H\Lambda^0(\mathrm{d},\Omega)} \leq C\Big(\underbrace{\inf_{\mathbf{v}_h\in\mathcal{W}^1(\mathcal{M})}\|\mathbf{a}-\mathbf{v}_h\|_{H\Lambda^1(\mathrm{d},\Omega)}}_{=O(h_\mathcal{M})} + \underbrace{\inf_{q_h\in\mathcal{W}^0(\mathcal{M})}\|p-q_h\|_{H\Lambda^0(\mathrm{d},\Omega)}}_{=O(h_\mathcal{M})}\Big), \quad (13.3.3.70)$$

For uniformly shape regular families of meshes

Corollary 13.3.3.71. Asymptotic Whitney finite element error estimate

Provided that $\mathbf{a}\in H^1\Lambda^1(\Omega)$, $\mathrm{d}_1\mathbf{a}\in H^1\Lambda^2(\Omega)$, $p\in H^2\Lambda^0(\Omega)$, *the asymptotic discretization error estimate*

$$\|\mathbf{a}-\mathbf{a}_h\|_{H\Lambda^1(\mathrm{d},\Omega)} + \|p-p_h\|_{H\Lambda^0(\mathrm{d},\Omega)} = O(h_\mathcal{M}) \quad \text{for} \quad h_\mathcal{M}\to 0 \qquad (13.3.3.72)$$

holds true for uniformly shape regular families of simplicial meshes $\mathcal{M}$ *with meshwidths* $h_\mathcal{M}$.