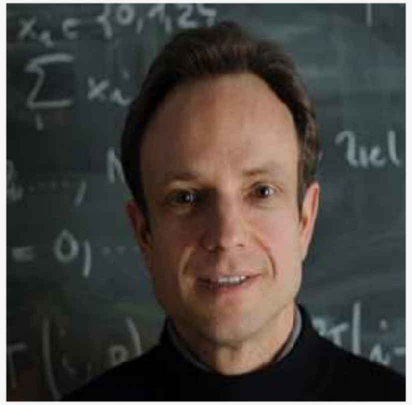
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 2.2: Principles of Galerkin Discretization



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Prerequisites.

- Fundamentals of linear algebra: bases, matrices, vectors

Dependency. Requires familiarity with linear variational problems [Lecture → Section 1.4] and quadratic minimization problems [Lecture → Section 1.2.3].

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

II. Finite Element Methods

target weak formulation of elliptic BVPs

- 2nd-order elliptic Dirichlet problem: ∞-dimensional function space

$$u \in H^1(\Omega),\ u = g \text{ on } \partial\Omega: \quad \int_\Omega (\alpha(x)\,\mathbf{grad}\,u(x))\cdot\mathbf{grad}\,v(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^1(\Omega), \qquad (1.4.2.4)$$

↳ uniformly positive

- 2nd-order elliptic Neumann problems:

$$u \in H_*^1(\Omega): \quad \int_\Omega (\alpha(x)\,\mathbf{grad}\,u)\cdot\mathbf{grad}\,v\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x + \int_{\partial\Omega} hv\,\mathrm{d}S \quad \forall v \in H_*^1(\Omega), \qquad (1.8.0.16)$$

Mission of FEM: Discretization = Conversion into a problem posed on a finite-dimensional space.

Variational boundary value problem	DISCRETIZATION →	System of a finite number of equations for (real) unknowns

2.2. Principles of Galerkin Discretization (GD)

Abstract LVP (→ Def 1.4.1.6) ↗ linear form, cont. w.r.t. $\|\cdot\|_a$

$$u \in V_0: \quad a(u,v) = \ell(v) \quad \forall v \in V_0$$

↳ energy space ↳ s.p.d. bilinear form → energy norm $\|\cdot\|_a$

(2)

2.2.1. GD : 1st Step

Replace the infinite-dimensional function space V_0 in the linear variational problem (2.2.0.2) with a finite-dimensional subspace $V_{0,h} \subset V_0$.

"h" ≙ tag for sth. finite

GD-I : $V_0 \longrightarrow V_{0,h} \subset V_0$, $\dim V_{0,h} < \infty$

Discrete (linear) variational problem (DVP):

$$u_h \in V_{0,h}: \quad \mathsf{a}(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_{0,h}\,. \tag{2.2.1.1}$$

the Galerkin solution

$V_{0,h} \subset V_0 \Rightarrow$ everything remains well-defined

(DVP) $\Leftrightarrow$ Discrete QMP

Continuous minimization problem $u = \operatorname{argmin}_{v \in V_0} J(v)$.	Ritz method $\longrightarrow$	*Discrete* minimization problem $u_h = \operatorname{argmin}_{v_h \in V_{0,h}} J(v_h)$.
$\Updownarrow$		$\Updownarrow$
Continuous variational problem $u \in V_0$: $\mathsf{a}(u,v) = \ell(v) \quad \forall v \in V_0$.	Galerkin disc. $\longrightarrow$	*Discrete* variational problem $u_h \in V_{0,h}$: $\mathsf{a}(u_h,v_h) = \ell(v_h) \quad \forall v_h \in V_{0,h}$.

Theory much easier in fin.-dim. setting

$\mathsf{a}(\cdot,\cdot)$ s.p.d $\Rightarrow$ solution of DVP unique

LA: In finite dim.: uniqueness $\Leftrightarrow$ existence

Theorem 2.2.1.5. Existence and uniqueness of solutions of discrete variational problems

If the bilinear form $\mathsf{a} : V_0 \times V_0 \mapsto \mathbb{R}$ *is symmetric and positive definite* (→ *Def.* 1.2.3.30) *and the linear form* $\ell : V_0 \mapsto \mathbb{R}$ *is continuous in the sense of*

$$\exists C_\ell > 0: \quad |\ell(u)| \leq C_\ell \|u\|_{\mathsf{a}} \quad \forall u \in V_0\,, \tag{1.2.3.43}$$

then the discrete variational problem (2.2.1.1) *has a unique Galerkin solution* $u_h \in V_{0,h}$ *that satisfies the energy estimate*

$$\|u_h\|_{\mathsf{a}} \leq \sup_{v_h \in V_{0,h}} \frac{|\ell(v_h)|}{\|v_h\|_{\mathsf{a}}} \leq C_\ell\,. \tag{2.2.1.6}$$

2.2.2: GD : 2nd Step

Toward a (linear) system of equation

Idea: ✦ choose (ordered) basis $\mathfrak{B}_h = \{b_h^1, \dots, b_h^N\}$, of $V_{0,h}$:

$$V_{0,h} = \operatorname{Span}\{\mathfrak{B}_h\} \qquad N := \dim V_{0,h}$$

✦ insert basis representation into the variational equation (2.2.1.1)

$$v_h \in V_{0,h} \quad \Rightarrow \quad v_h = \nu_1 b_h^1 + \dots + \nu_N b_h^N\,, \quad \nu_i \in \mathbb{R}\,, \tag{2.2.2.1}$$

$$u_h \in V_{0,h} \quad \Rightarrow \quad u_h = \mu_1 b_h^1 + \dots + \mu_N b_h^N\,, \quad \mu_i \in \mathbb{R}\,. \tag{2.2.2.2}$$

③ Now plug basis expansions for u_h, v_h into DVP and use (bi-) linearity ; (∗)

DVP: $u_h \in V_{0,h}$: $\mathsf{a}(u_h, v_h) = \ell(v_h)$ $\forall v_h \in V_{0,h}$. (2.2.1.1)

(∗) $\Updownarrow$ $\left[\begin{array}{l} u_h = \mu_1 b_h^1 + \cdots + \mu_N b_h^N, \mu_i \in \mathbb{R} \\ v_h = \nu_1 b_h^1 + \cdots + \nu_N b_h^N, \nu_i \in \mathbb{R} \end{array} \right]$

$$\sum_{k=1}^{N}\sum_{j=1}^{N} \mu_k \nu_j \mathsf{a}(b_h^k, b_h^j) = \sum_{j=1}^{N} \nu_j \ell(b_h^j) \quad \forall \nu_1, \dots, \nu_N \in \mathbb{R},$$

$\Updownarrow$ ← equivalence !

$$\sum_{j=1}^{N} \nu_j \left(\sum_{k=1}^{N} \mu_k \mathsf{a}(b_h^k, b_h^j) - \ell(b_h^j) \right) = 0 \quad \forall \nu_1, \dots, \nu_N \in \mathbb{R},$$

LA fact : linear form vanishes $\forall$ basis vectors $\Leftrightarrow$ linear form $\equiv 0$

$\Updownarrow$ (∗)

$$\sum_{k=1}^{N} \mu_k \mathsf{a}(b_h^k, b_h^j) = \ell(b_h^j) \quad \text{for } j = 1, \dots, N .$$

$\Updownarrow$ $[\vec{\mu} = (\mu_1, \dots, \mu_N)^\top \in \mathbb{R}^N]$

Galerkin matrix

$$\boxed{\mathbf{A}\vec{\mu} = \vec{\varphi}}, \quad \text{with} \quad \mathbf{A} = \left[\mathsf{a}(b_h^k, b_h^j)\right]_{j,k=1}^{N} \in \mathbb{R}^{N,N}, \quad \vec{\varphi} = \left[\ell(b_h^j)\right]_{j=1}^{N} .$$

A linear system of equations (LSE)

Q : After having solved $A\vec{\mu} = \vec{\varphi}$, how do we get u_h ?

$$u_h = \sum_{j=1}^{N} \mu_j b_h^j \in V_{0,h}$$

A lot of freedom in choosing $\mathcal{B}_h$:

Theorem 2.2.2.7. Independence of Galerkin solution of choice of basis

The choice of the basis $\mathfrak{B}_h$ has no impact on the (set of) Galerkin solutions u_h of (2.2.1.1).

2.2.3. Galerkin Matrices

$$(A)_{j,k} = a(b_h^k, b_h^j), \quad 1 \le k, j \le N := \dim V_{0,h}$$

Corollary 2.2.3.1.

(2.2.1.1) *has unique solution* $\Leftrightarrow$ $\mathbf{A}$ *nonsingular (invertible)*

How do A and $\vec{\varphi}$ depend on $\mathcal{B}$?

Two bases : $\mathcal{B} = \{ b_h^j \}_{j=1}^N \to A \in \mathbb{R}^{N,N}$

$\tilde{\mathcal{B}} = \{ \tilde{b}_h^j \}_{j=1}^N \to \tilde{A} \in \mathbb{R}^{N,N}$

$$\tilde{b}_h^j = \sum_{k=1}^{N} s_{jk} b_h^k, \quad s_{jk} \in \mathbb{R}, \quad S = [s_{jk}]_{j,k=1}^{N}$$

basis transformation matrix

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$$(\widetilde{A})_{ij} = a(\tilde{b}_h^j, \tilde{b}_h^i)$$
$$= a\Big(\sum_{k=1}^{N} s_{jk} b_h^k, \sum_{\ell=1}^{N} s_{i\ell} b_h^\ell\Big)$$

use bi-linearity
$$= \sum_k \sum_\ell s_{jk} s_{i\ell} \underbrace{a(b_h^k, b_h^\ell)}_{(A)_{\ell,k}} = (SAS^T)_{ij}$$

Same arguments:

$$\big(\vec{\tilde{\varphi}}\big)_j = \ell(\tilde{b}_h^j) = \ell\Big(\sum_{k=1}^{N} s_{jk} b_h^k\Big) = \sum_{k=1}^{N} s_{jk}\ell(b_h^k) = \sum_{k=1}^{N} s_{jk}(\vec{\varphi})_k = (\mathbf{S}\vec{\varphi})_j,$$

Lemma 2.2.3.2. Effect of change of basis on Galerkin matrix

Consider (2.2.1.1) *and two bases of* $V_{0,h}$,

$$\mathfrak{B}_h := \{b_h^1, \dots, b_h^N\} \quad , \quad \widetilde{\mathfrak{B}}_h := \{\tilde{b}_h^1, \dots, \tilde{b}_h^N\} ,$$

related by the basis transformation matrix $\mathbf{S}$ *according to*

$$\tilde{b}_h^j = \sum_{k=1}^{N} s_{jk} b_h^k \quad \text{with} \quad \mathbf{S} = [s_{jk}]_{j,k=1}^{N} \in \mathbb{R}^{N,N} \text{ regular.} \tag{2.2.3.3}$$

Then the Galerkin matrices $\mathbf{A}, \widetilde{\mathbf{A}} \in \mathbb{R}^{N,N}$, *the right hand side vectors* $\vec{\varphi}, \vec{\tilde{\varphi}} \in \mathbb{R}^N$, *and the coefficient vectors* $\vec{\mu}, \vec{\tilde{\mu}} \in \mathbb{R}^N$, *respectively, satisfy*

$$\widetilde{\mathbf{A}} = \mathbf{SAS}^\top \quad , \quad \vec{\tilde{\varphi}} = \mathbf{S}\vec{\varphi} \quad , \quad \vec{\tilde{\mu}} = \mathbf{S}^{-\top}\vec{\mu} \, . \tag{2.2.3.4}$$

$\Downarrow$

Definition 2.2.3.5. Congruent matrices

Two matrices $\mathbf{A} \in \mathbb{R}^{N,N}$, $\mathbf{B} \in \mathbb{R}^{N,N}$, $N \in \mathbb{N}$, are called congruent, if there is a regular matrix $\mathbf{S} \in \mathbb{R}^{N,N}$ such that $\mathbf{B} = \mathbf{SAS}^T$.

Any regular $S \in \mathbb{R}^{N,N}$ can serve as basis tf. matrix

$\triangleright$ Galerkin matrix for a given DVP can be any matrix from a congruence class

Q: What properties are shared by all Galerkin matrices for a DVP?

- Regularity = Invertibility
- Symmetry
- Pos. def.

$\Big\} \Longleftrightarrow a(\cdot,\cdot)$ s.p.d. on $V_{0,h}$

Take-home message:

Thm. 2.2.2.7: the choice of $\mathfrak{B}_h$ does not affect u_h (assuming exact arithmetic) $\Rightarrow$ No impact on discretization error !

But: Key properties (e.g., conditioning, sparsity) of the Galerkin matrix crucially depend on $\mathfrak{B}_h$!

- The choice of trial/test spaces $V_{0,h}$ alone determines the quality of the Galerkin solution.
- The choice of basis $\mathfrak{B}_h$ determines how well (stably, efficiently) we can compute the Galerkin solution.

⑤ Review questions 2.2.3.10 :*,**

A:

Described the **first step** of the Galerkin discretization of a linear variational problem. What does it yield?

B:

The Galerkin discretization of a linear variational problem leads to a linear system of equations. What determines the size of its Galerkin matrix?

C:

Give simple *necessary and sufficient* conditions for existence and uniqueness of solutions of a linear system of equations arising from the Galerkin discretization of a linear variational problem.

D:

Write $\mathbf{A}\vec{\mu} = \vec{\varphi}$ for the square linear system of equations produced by the Galerkin discretization of a linear variational problem posed on the vector space V_0 and using the basis $\mathfrak{B} := \{b_h^1, \dots, b_h^N\}$ of the discrete trial/test space $V_{0,h}$, $N := \dim V_{0,h}$. Now we switch to the rescaled basis $\widetilde{\mathfrak{B}} := \{b_h^1, 2b_h^2, 3b_h^3, \dots, Nb_h^N\}$. What linear system will we get?

* Try to answer the review questions shortly after having studied the topic and without looking up information in course documents or online.

** Additional review questions may be given in lecture document.

E:

Let $\mathbf{A}\vec{\mu} = \vec{\varphi}$, $\mathbf{A} \in \mathbb{R}^{N,N}$, $\vec{\varphi} \in \mathbb{R}^N$, $N \in \mathbb{N}$, be the linear system of equations obtained by the Galerkin discretization of a linear variational problem (LVP)

$$u \in V_0: \quad \mathsf{a}(u, v) = \ell(v) \quad \forall v \in V_0 ,$$

using a finite-dimensional trial/test space $V_{0,h} \subset V_0$ equipped with basis $\mathfrak{B} := \{b_h^1, \dots, b_h^N\}$. Which linear system do we get when performing the Galerkin discretization of the same LVP but dropping every other basis function, that is, using the discrete trial/test space spanned by $\{b_h^1, b_h^3, \dots, b_h^{N-1}\}$ (for even N).

F:

Which properties of a Galerkin matrix do not depend on the choice of basis for the discrete trial and test space?

G:

Let $\mathbf{A} \in \mathbb{R}^{N,N}$ be the Galerkin matrix obtained by the Galerkin discretization of a bilinear form $\mathsf{a}(\cdot,\cdot)$. Given $\vec{\mu}, \vec{\eta} \in \mathbb{R}^N$, express the number $\vec{\mu}^\top \mathbf{A}\vec{\eta}$ through $\mathsf{a}(\cdot,\cdot)$.

H:

As regards Galerkin discretization of a linear variational problem, which properties of the right-hand-side vector do not depend on the choice of basis for the discrete trial/test space?

I :

On a vector space V_0 we consider a linear variational problem

$$u \in V_0\colon \quad \mathsf{a}(u,v) = \ell(v) \quad \forall v \in V_0 \,,$$

with a *symmetric positive definite* bilinear form a and an $\|\cdot\|_{\mathsf{a}}$-bounded linear form $\ell : V_0 \to \mathbb{R}$. Show that for every finite-dimensional subspace $V_{0,h} \subset V_0$ there exists a basis $\mathfrak{B} \subset V_{0,h}$ such that the associated Galerkin matrix for $\mathsf{a}(\cdot,\cdot)$ is the indentity matrix.

Hint. Use the following fundamental result from linear algebra:

Theorem 0.3.1.26. Real diagonalization of symmetric matrices

For every symmetric matrix $\mathbf{A} \in \mathbb{R}^{n,n}$, that is, $\mathbf{A}^\top = \mathbf{A}$, we can find a diagonal matrix $D \in \mathbb{R}^{n,n}$ and an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{n,n}$ such that

$$\mathbf{Q}^\top \mathbf{A} \mathbf{Q} = \mathbf{D} \,. \tag{2.2.3.11}$$

For a symmetric positive definite matrix $\mathbf{A}$ the non-zero entries of the diagonal matrix $\mathbf{D}$ are all positive.