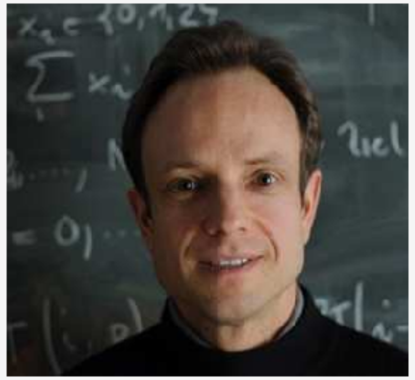
1

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 2.4: Case Study: Triangular Linear FEM in Two Dimensions (I)

Prof. R. Hiptmair, SAM, ETH Zurich



Date: February 21, 2019

Prerequisites.

- Weak formulation of linear second-order elliptic BVP
- Understanding of Galerkin discretization

Dependency. Requires [Lecture → 1.5.3] and [Lecture → Section 2.2]. Familiarity with [Lecture → Section 2.3] is beneficial.

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

II. Finite Element Methods (FEM)

2.4. Case Study: Triangular Linear FEM in 2D

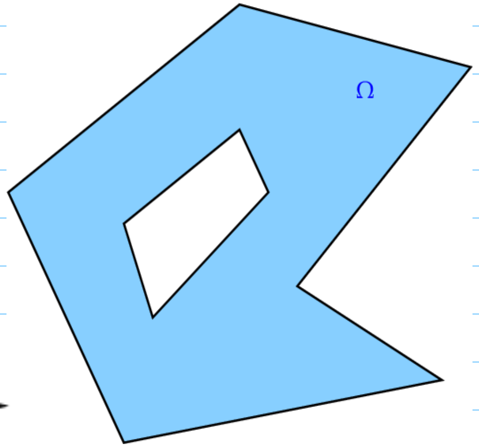
Neumann model problem

$$u \in H^1(\Omega): \quad \int_\Omega \boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u \cdot \mathbf{grad}\,v + c(\boldsymbol{x})\,u\,v\,\mathrm{d}\boldsymbol{x} = \int_\Omega f v\,\mathrm{d}\boldsymbol{x} \quad \forall v \in H^1(\Omega)\,, \qquad (2.1.2.2)$$

$\Updownarrow \leftarrow$ see Section 1.5

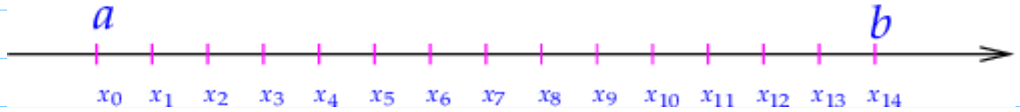
strong form, BVP:
$$\begin{aligned} -\operatorname{div}(\boldsymbol{\alpha}(\boldsymbol{x})\,\mathbf{grad}\,u) + c(\boldsymbol{x})u &= f \quad \text{in } \Omega \subset \mathbb{R}^2 \\ \mathbf{grad}\,u \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}$$

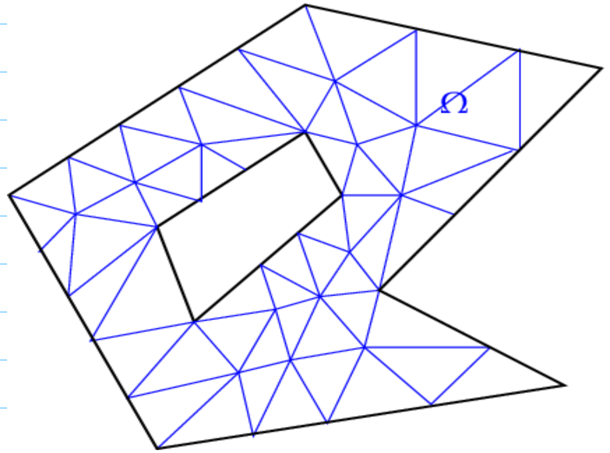
Admissible domains : polygons ▷
(§2.4.0.1)



2.4.1 2D Meshes : Triangulations

Recall 1D: partition of $\Omega =]a,b[$:





A **triangulation** $\mathcal{M}$ of Ω satisfies

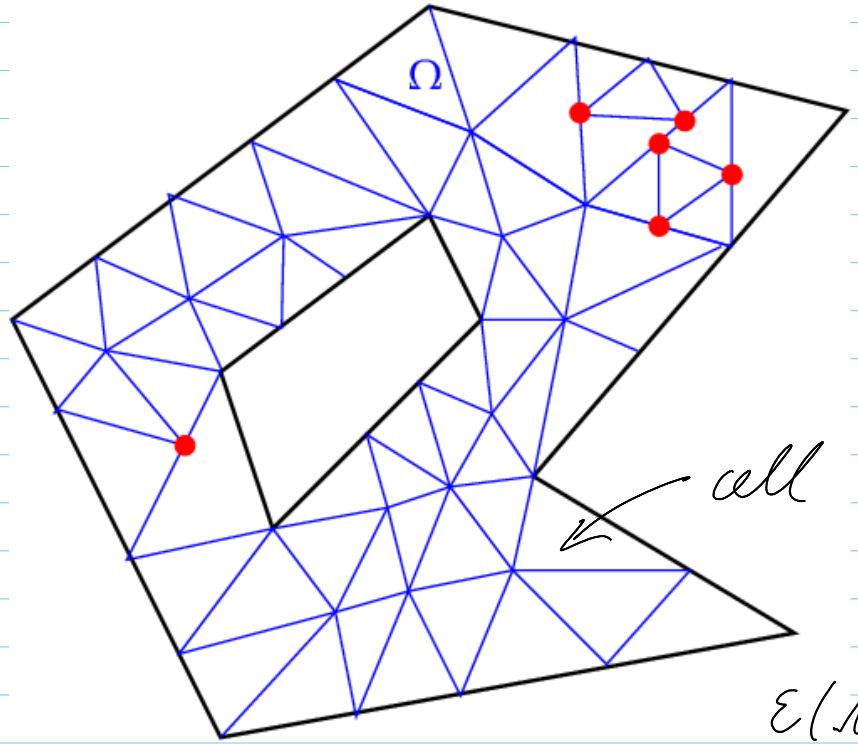
(i) $\mathcal{M} = \{K_i\}_{i=1}^M$, $M \in \mathbb{N}$, $K_i \hat{=}$ open triangle

(ii) disjoint interiors: $i \neq j \Rightarrow K_i \cap K_j = \emptyset$

(iii) tiling/partition property: $\bigcup_{i=1}^{M} \overline{K}_i = \overline{\Omega}$

(iv) intersection $\overline{K}_i \cap \overline{K}_j$, $i \neq j$, is
- either $\emptyset$
- or an edge of both triangles
- or a vertex of both triangles

2



$\triangleleft$ illegal triangulation

$\bullet \stackrel{\wedge}{=}$ hanging nodes

cell

$\mathcal{E}(\mathcal{M}) \stackrel{\wedge}{=}$ set of edges

Notations: $\mathcal{M} = \{K_1, \dots, K_M\}$, $M \in \mathbb{N}$

nodes $\mathcal{V}(\mathcal{M}) = \{x^1, \dots, x^N\}$, $N \in \mathbb{N}$

§2.4.4.1

Data structures:

$M \times 3$ matrix of vertex indices

$N \times 2$-matrix: rows contain node coordinates

C++ code 2.4.1.2: Class handling planar triangular mesh ➜ GITLAB

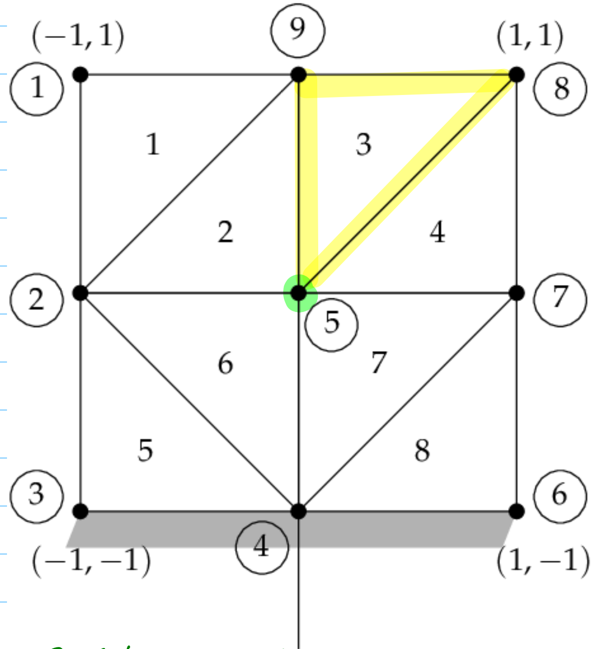
```
// Matrix containing vertex coordinates of a triangle
using TriGeo_t = Eigen::Matrix<double, 2, 3>;
struct TriaMesh2D {
  // Constructor: reads mesh data from file, whose name is passed
  TriaMesh2D(const std::string&); //
  virtual ~TriaMesh2D(void) {}

  // Retrieve coordinates of vertices of a triangles as rows of a 3x2
     matrix
  TriGeo_t getVtCoords(std::size_t) const;

  // Data members describing geometry and topolgy
  Eigen::Matrix<double, Eigen::Dynamic, 2> Coordinates;
  Eigen::Matrix<int, Eigen::Dynamic, 3> Elements;
};
```

implies an indexing of nodes

EXAMPLE 2.4.1.4 (Internal array representation of 2D triangular mesh) We make explicit the contents of the **Coordinates** and **Elements** matrices, data members of **TriaMesh2D**, for the special triangulation of the square $\Omega =]-1,1[^2$ drawn in Fig. 52.



i	$(x_i)_1$	$(x_i)_2$
1	-1	1
2	-1	0
3	-1	-1
4	0	-1
5	0	0
6	1	-1
7	1	0
8	1	1
9	0	1

Matrix **Coordinates**

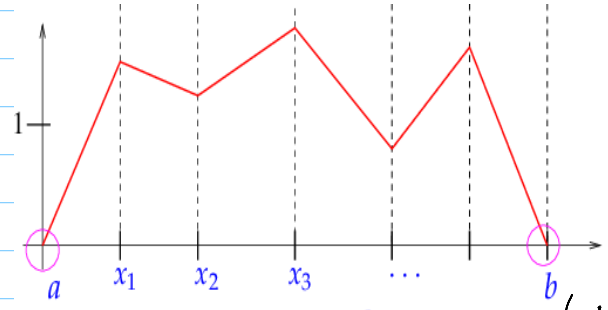
K_j	Vertex indices		
1	1	2	9
2	2	5	9
3	5	8	9
4	5	7	8
5	3	4	2
6	4	5	2
7	4	7	5
8	4	6	7

Matrix **Elements**

2.4.2. Linear Finite Element Space

§2.4.2.1:

Recall 1D:



⇑ function $\in \mathcal{S}^0_{1,0}(\mathcal{M}) \subset H^1_0(]0,1[)$

1D linear finite element trial space on mesh $\mathcal{M} := \{]x_{j-1}, x_j[: j = 1, \dots, M\}$ of $\Omega :=]a,b[\subset \mathbb{R}$:

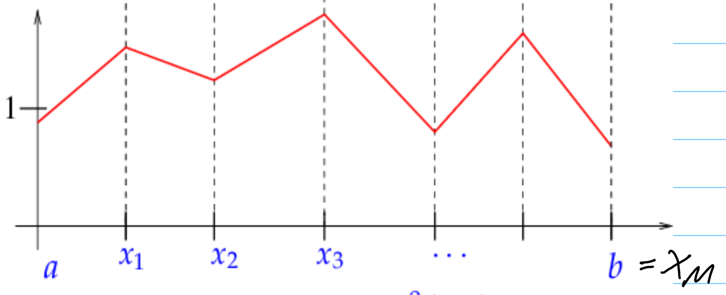
$$\mathcal{S}^0_{1,0}(\mathcal{M}) := \left\{ \begin{array}{l} v \in C^0([a,b]) : v_{|[x_{i-1},x_i]} \text{ linear}, \\ i = 1, \dots, M, v(a) = v(b) = 0 \end{array} \right\}$$

No b.c.:

Full 1D linear finite element space on mesh $\mathcal{M} := \{]x_{j-1}, x_j[: j = 1, \dots, M\}$ of $]a,b[\subset \mathbb{R}$:

$$\mathcal{S}^0_1(\mathcal{M}) := \{ v \in C^0([a,b]) : v_{|[x_{i-1},x_i]} \text{ linear } \forall i \} \quad (2.4.2.2)$$

$\dim \mathcal{S}^0_1(\mathcal{M}) = M+1$



⇑ function $\in \mathcal{S}^0_1(\mathcal{M})$

Linear function in 1D: $x \to \alpha + \beta x$, $\alpha, \beta \in \mathbb{R}$

Linear function in 2D: $\underline{x} \in \mathbb{R}^2 \to \alpha + \underline{\beta} \cdot \underline{x}$, $\alpha \in \mathbb{R}$, $\underline{\beta} \in \mathbb{R}^2$

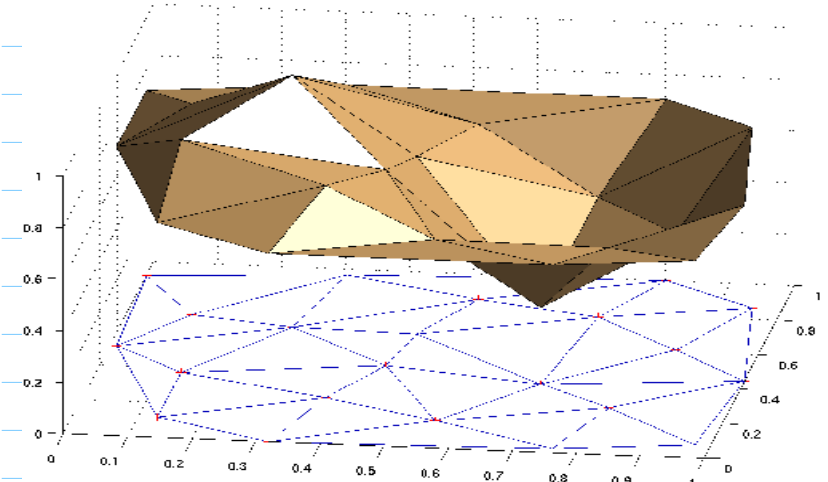
	$d=1$	$d=2$
Grid/mesh cells:	intervals $]x_{i-1}, x_i[$, $i = 1, \dots, M$	triangles K_i, $i = 1, \dots, M$
Linear functions:	$x \in \mathbb{R} \mapsto \alpha + \beta \cdot x$, $\alpha, \beta \in \mathbb{R}$	$\boldsymbol{x} \in \mathbb{R}^2 \mapsto \alpha + \boldsymbol{\beta} \cdot \boldsymbol{x}$, $\alpha \in \mathbb{R}$, $\boldsymbol{\beta} \in \mathbb{R}^2$

Tentative: parallel to 1D

$$\boxed{V_{0,h} = \mathcal{S}_1^0(\mathcal{M}) := \left\{ v \in C^0(\overline{\Omega}) : \forall K \in \mathcal{M} : \begin{array}{l} v_{|K}(\boldsymbol{x}) = \alpha_K + \boldsymbol{\beta}_K \cdot \boldsymbol{x}, \\ \alpha_K \in \mathbb{R}, \boldsymbol{\beta}_K \in \mathbb{R}^2, \boldsymbol{x} \in K \end{array} \right\} \subset H^1(\Omega)}$$

see Thm. 1.3.4.23

Space of $\mathcal{M}$-p.w. linear continuous function on Ω

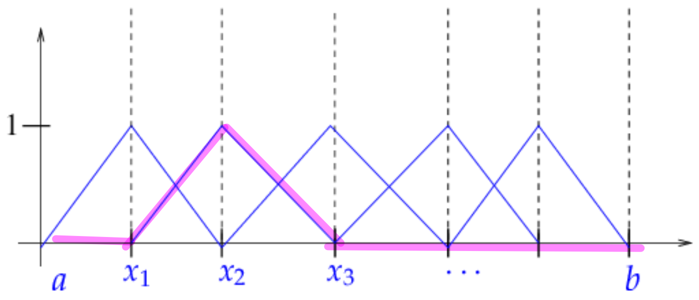


$\mathcal{S}_1^0(\mathcal{M}) \neq \{0\}$

$\triangle$ "Visual proof"

2.4.3. Nodal basis functions

1D: 1D "tent functions" $\triangleright$

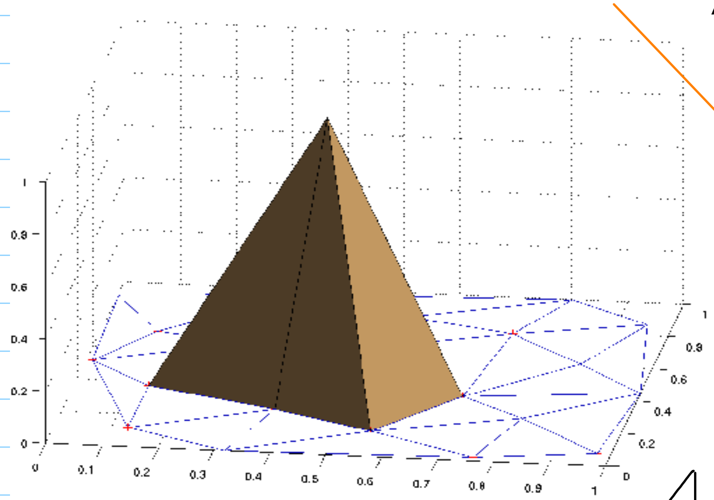


$$\mathfrak{B} = \{b_h^1, \dots, b_h^{M-1}\}, \qquad (2.4.3.1)$$

$$b_h^j(x_i) = \delta_{ij} := \begin{cases} 1 & \text{, if } i = j, \\ 0 & \text{, if } i \neq j, \end{cases} \qquad (2.4.2.2)$$

cardinal basis property

$b_h^x \in \mathcal{S}_1^0(\mathcal{M})$



Idea: *define* (**?**) basis function $b_h^{\boldsymbol{x}}$, $\boldsymbol{x} \in \mathcal{V}(\mathcal{M})$, by "nodal conditions"

$$b_h^{\boldsymbol{x}}(\boldsymbol{y}) = \begin{cases} 1 & \text{, if } \boldsymbol{y} = \boldsymbol{x}, \\ 0 & \text{, if } \boldsymbol{y} \in \mathcal{V}(\mathcal{M}) \setminus \{\boldsymbol{x}\}. \end{cases} \qquad (2.4.3.2)$$

Is this possible ?

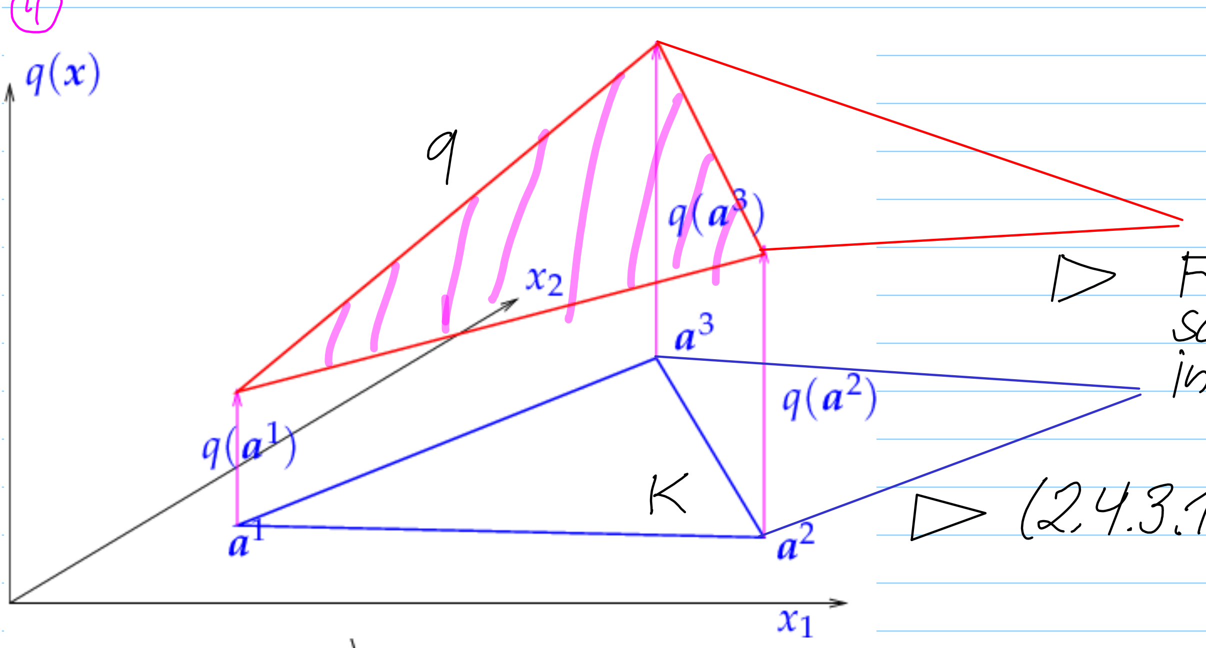
$\triangle$ "Visual proof of existence"

§ 2.4.3.3: On a triangle K with vertices $\underline{a}^1, \underline{a}^2, \underline{a}^3$ look for a

$$b(\underline{x}) = \alpha + \underline{\beta} \cdot \underline{x}, \quad \alpha \in \mathbb{R}, \ \underline{\beta} \in \mathbb{R}^2$$

such that $b(\underline{a}^j) = q_j \in \mathbb{R}$

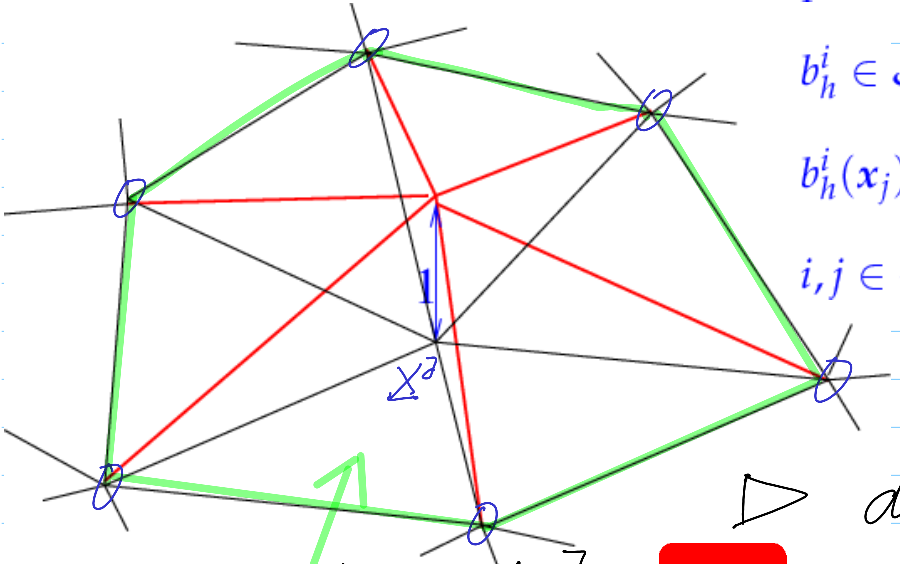
$$\det = 2|K| \neq 0 ; \left\{ \begin{bmatrix} 1 & a_1^1 & a_2^1 \\ 1 & a_1^2 & a_2^2 \\ 1 & a_1^3 & a_2^3 \end{bmatrix} \right. \begin{bmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$



q is linear on all edges

▷ Fixed by prescribing values in endpoints

▷ (2.4.3.1) ⇒ C^0-continuity



$$b_h^i \in \mathcal{S}_1^0(\mathcal{M}) \,,$$

$$b_h^i(\boldsymbol{x}_j) = \begin{cases} 1 & \text{, if } i = j \,, \\ 0 & \text{else,} \end{cases} \qquad (2.4.3.5)$$

$$i,j \in \{1,\dots,N\} \,.$$

◁ 2D tent functions

▷ $\dim \mathcal{S}_1^0(\mathcal{M}) = \#\mathcal{V}(\mathcal{M}) = N$

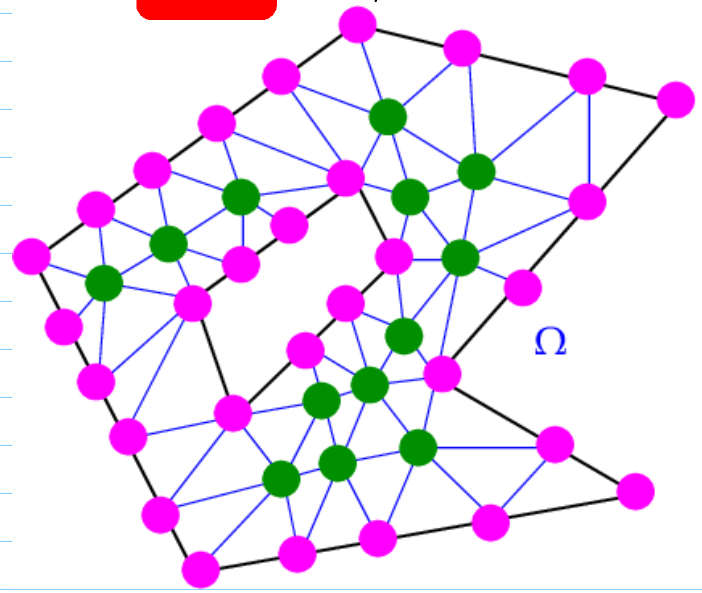
Q: Why linearly independent?

▷ $\mathcal{B}_h = \{b_h^i\}_{i=1}^N$

↑ associated with x^i

§ 2.4.3.7 (Linear finite element space $\subset H_0^1(\Omega)$)

Q: How to modify $\mathcal{S}_1^0(\mathcal{M})$ to obtain $\mathcal{S}_{1,0}^0(\mathcal{M}) \subset \mathcal{S}_1^0(\mathcal{M})$, $\mathcal{S}_{1,0}^0(\mathcal{M}) \subset H_0^1(\Omega)$?



◁ "Location" of nodal basis functions: (mesh $\mathcal{M}$ → Fig. 49)

●, ● → nodal basis functions of $\mathcal{S}_1^0(\mathcal{M})$

● → nodal basis functions of $\mathcal{S}_{1,0}^0(\mathcal{M})$

$\mathcal{S}_{1,0}^0(\mathcal{M}) = \mathrm{Span}\{b_h^j : x^j \notin \partial\Omega\}$

2.4.4. Sparse Galerkin Matrix

Notion 2.4.4.3. Sparse matrix

$\mathbf{A} \in \mathbb{K}^{m,n}$, $m,n \in \mathbb{N}$, is sparse, if

$$\mathrm{nnz}(\mathbf{A}) := \#\{(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\} \colon a_{ij} \neq 0\} \ll mn \,.$$

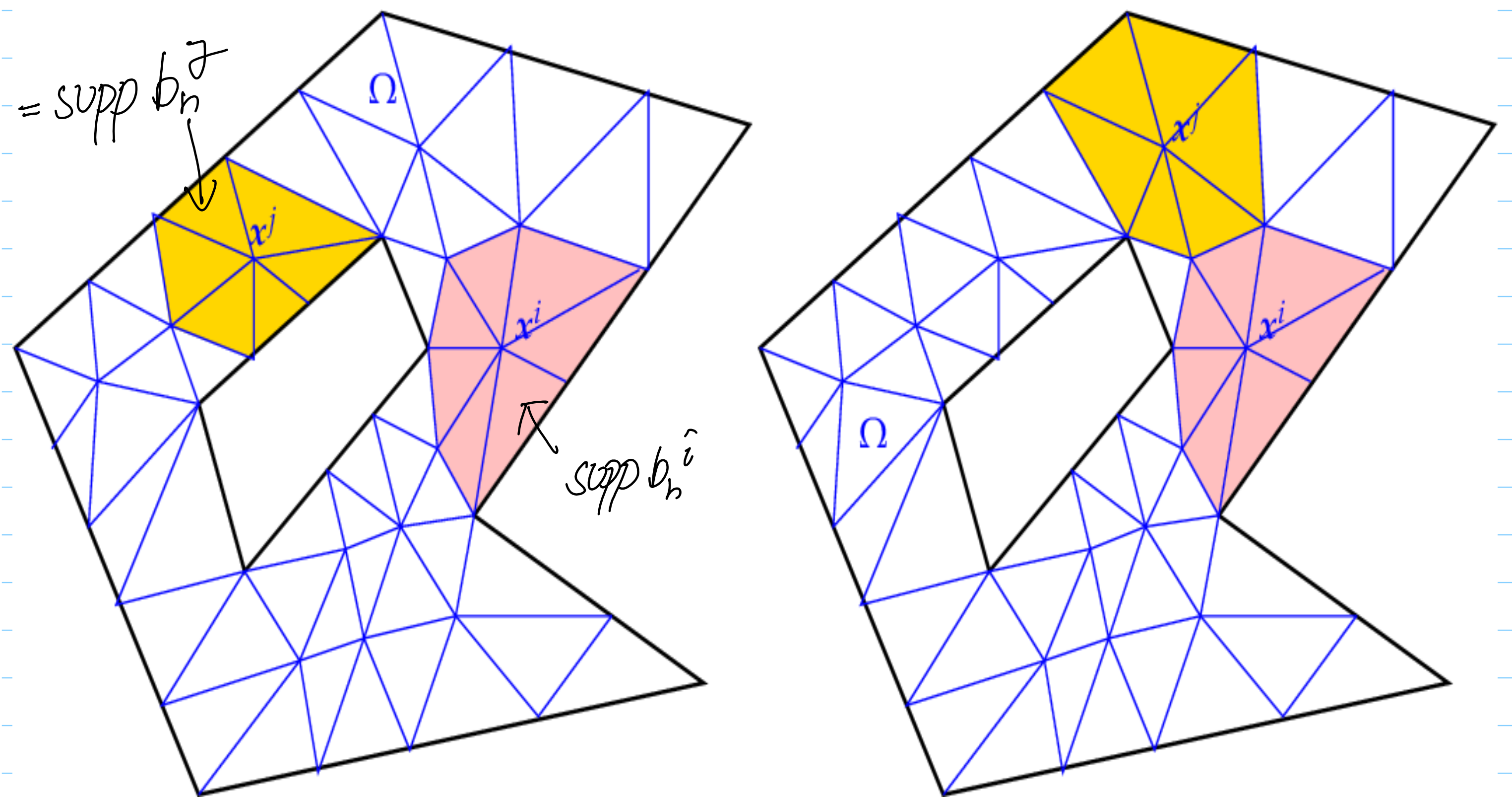
"Almost all entries = 0"

Recall formula for entries of Galerkin matrix

$$(A)_{k,l} = a(b_h^l, b_h^k) \,, \quad 1 \le k,l \le N$$

Here

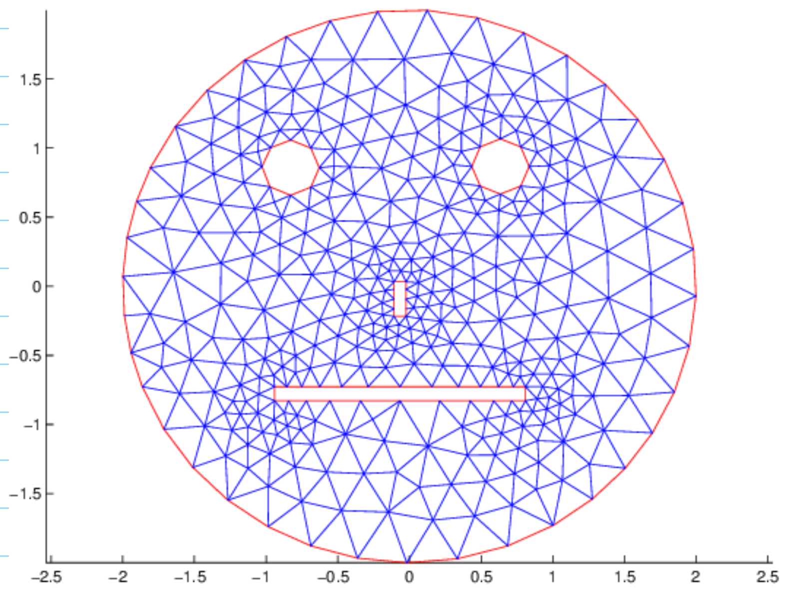
$$\mathsf{a}(u,v) := \int_{\Omega} (\alpha(\mathbf{x})\,\mathbf{grad}\,u)\cdot\mathbf{grad}\,v + c(\mathbf{x})u\,v\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} h\,v\,\mathrm{d}S\,,\quad u,v\in H^1(\Omega)\,. \tag{2.4.4.1}$$



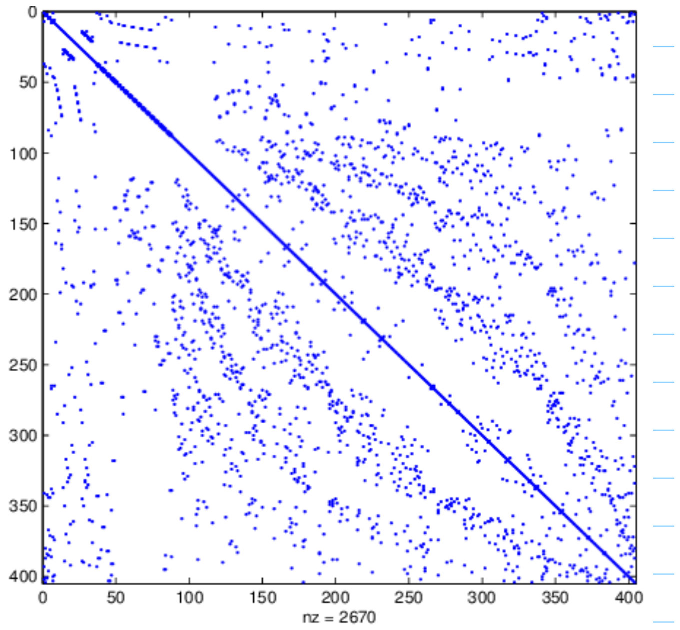
$$\boxed{\left\{\begin{array}{c}\text{Nodes } \mathbf{x}_i,\mathbf{x}_j\in\mathcal{V}(\mathcal{M})\\ \text{not connected by an edge}\end{array}\right\} \Leftrightarrow \underbrace{\mathrm{Vol}(\mathrm{supp}(b_h^i)\cap\mathrm{supp}(b_h^j))=0}_{\text{No overlap of supports}}\Bigg\} \Rightarrow (\mathbf{A})_{ij}=0\,.}$$

In particular: $\mathrm{nnz}(A) \leq \underbrace{\#\mathcal{V}(\mathcal{M})}_{\text{diagonal entry}} + 2\cdot\underbrace{\#\mathcal{E}(\mathcal{M})}_{\text{off-diagonal entries}}$

Ex 2.4.4.4 (Sparse Galerkin matrices)



Triangular mesh $\mathcal{M}$

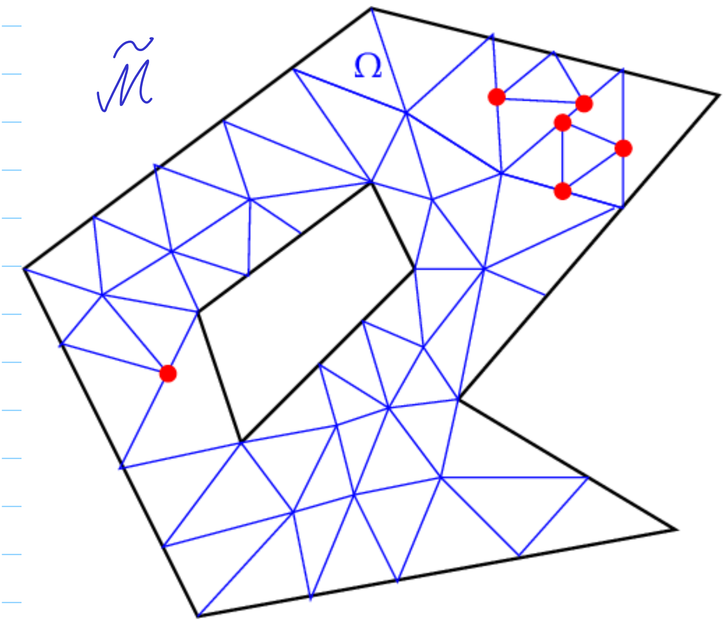


Resulting sparsity pattern of Galerkin matrix

⑥ # Review questions 2.4.6.13

[Answer without consulting any notes]

A :



The mesh $\widetilde{\mathcal{M}}$ displayed beside contains a few hanging nodes marked with $\bullet$.

Sketch a triangulation (without hanging nodes) with the same nodes as that mesh $\widetilde{\mathcal{M}}$.

B :

Chop up a square $\Omega \subset \mathbb{R}^2$ into n^2 congruent small squares and create a triangular mesh $\mathcal{M}$ of Ω by splitting each small square along parallel diagonals. What is $\dim \mathcal{S}_1^0(\mathcal{M})$ and $\dim \mathcal{S}_{1,0}^0(\mathcal{M})$ in terms of n?

C :

Let $\mathcal{M}$ be a triangulation of a polygonal domain. Describe a modification of the space $\mathcal{S}_1^0(\mathcal{M})$ of $\mathcal{M}$-piecewise linear continuous functions that yields a subspace $\mathcal{S}_{1,0}^0(\mathcal{M}) \subset \mathcal{S}_1^0(\mathcal{M})$ of *maximal dimension* such that $\mathcal{S}_{1,0}^0(\mathcal{M}) \subset H_0^1(\Omega)$.

D :

Prove that the barycentric coordinate functions for any non-degenerate planar triangle $K \subset \mathbb{R}^2$ are linearly independent.

E :

For the domain and mesh from Question (**Q**2.4.6.13.B) determine the maximal number of non-zerp entries of the Galerkin matrix obtained when discretizing (2.1.2.2) with trial and test space $\mathcal{S}_{1,0}^0(\mathcal{M})$ (sharp bound).

This list of review questions may not be complete. Additional review questions may be provided in the lecture document.