ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 2.6: Lagrangian Finite Element Methods

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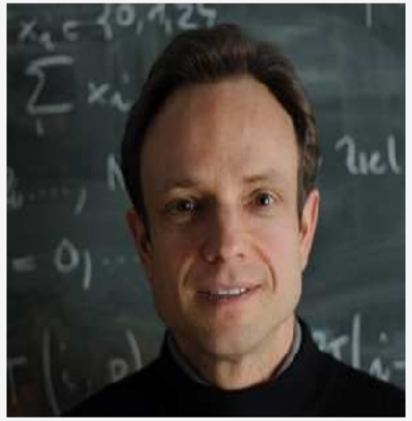


Date: February 22, 2019

Prerequisites.

- Calculus: polynomials and polynomial interpolation
- Galerkin discretization

Dependency. Relies on [Lecture → Section 2.2] and [Lecture → Section 2.5].

II. Finite Element Methods (FEM)

2.6. Lagrange FEM

use FE spaces $\subset H^1(\Omega)$, $\mathcal{M}$-p.w. polynomials

2.6.1. Simplicial Lagrangian FEM

↳ 2D: triangular, 3D: tetrahedral

▷ Generalize $\mathcal{S}_1^0(\mathcal{M})$

Definition 2.6.1.1. Simplicial Lagrangian finite element spaces

The space of **p-th degree Lagrangian finite element** functions on simplicial mesh $\mathcal{M}$ is defined as

$$\mathcal{S}_p^0(\mathcal{M}) := \{v \in C^0(\overline{\Omega}) : v_{|K} \in \mathcal{P}_p(K) \quad \forall K \in \mathcal{M}\}\,.$$

Nontrivial? → Specifying basis functions through demanding a cardinal basis property ($p=1$, w.r.t. point set $\mathcal{V}(\mathcal{M})$)

$$b_h^i(\boldsymbol{p}_j) = \delta_{ij}\,, \quad 1 \le i,j \le N := \dim \mathcal{S}_p^0(\mathcal{M})$$

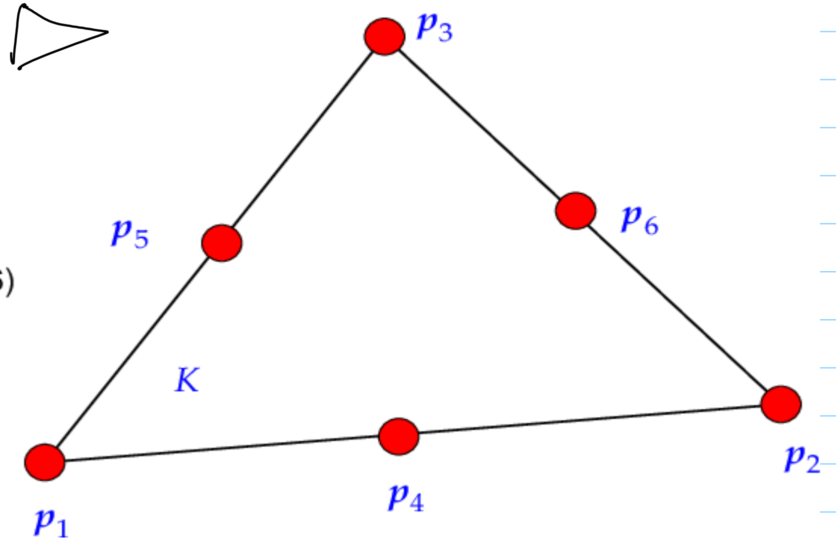
interpolation nodes

Examples: $d=2$, $p=2$ ($\leftrightarrow$ quadratic Lagr. FEM)

② (1) $\mathcal{M} = \{K\}$, $\mathcal{S}_p^0(\mathcal{M}) = \mathcal{P}_2(\mathbb{R}^2)$

↳ dim = 6

Interpolation nodes on K ▷



$$
\begin{aligned}
b_K^1 &= (2\lambda_1 - 1)\lambda_1 \,, \\
b_K^2 &= (2\lambda_2 - 1)\lambda_2 \,, \\
b_K^3 &= (2\lambda_3 - 1)\lambda_3 \,, \\
b_K^4 &= 4\lambda_1\lambda_2 \,, \\
b_K^5 &= 4\lambda_2\lambda_3 \,, \\
b_K^6 &= 4\lambda_1\lambda_3 \,.
\end{aligned}
\quad \in \mathcal{P}_2 \qquad (2.6.1.6)
$$

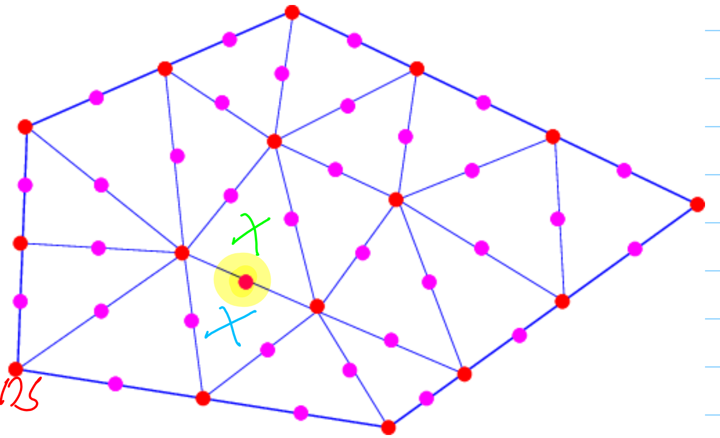
$b_K^j(p_i) = \delta_{ij}$

- General triangular mesh $\mathcal{M}$ → gluing

We fix the set of interpolation nodes

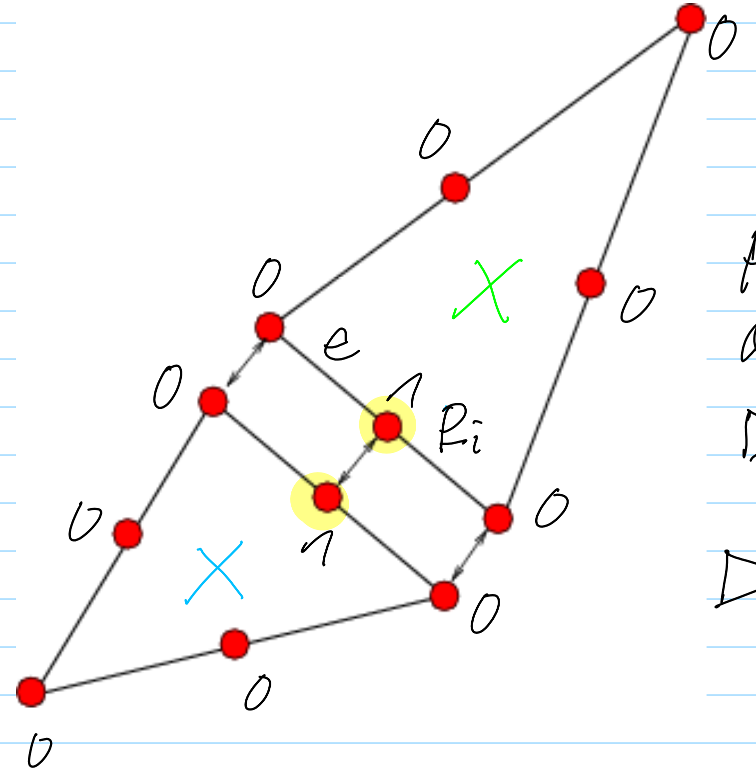
$\mathcal{N} := \mathcal{V}(\mathcal{M}) \cup \{\text{midpoints of edges}\}$,

$\mathcal{N} = \{p_1, \dots, p_N\}$ (ordered) .



▷ (2.6.1.6) = local shape functions

Big issue: $b_h^i \in C^0(\overline{\Omega})$?

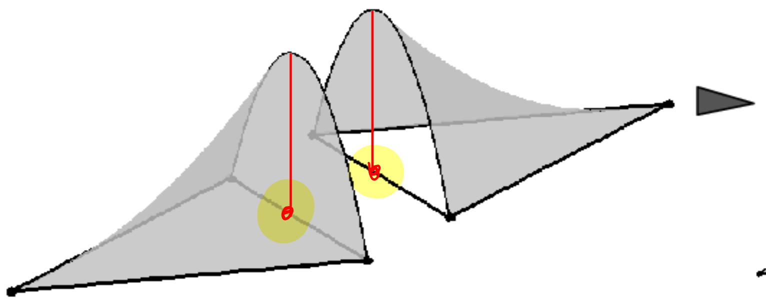


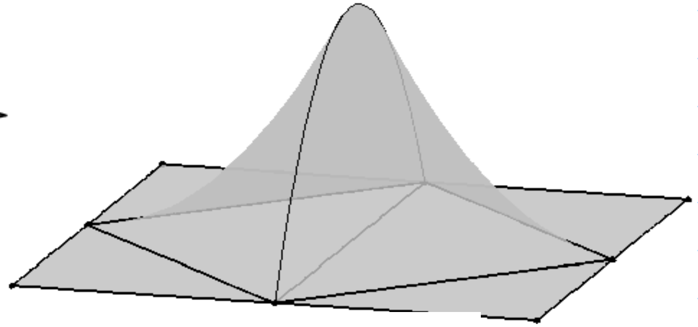
b_h^i is a 1D quadratic polynomial along edge e & fixed in 3 points

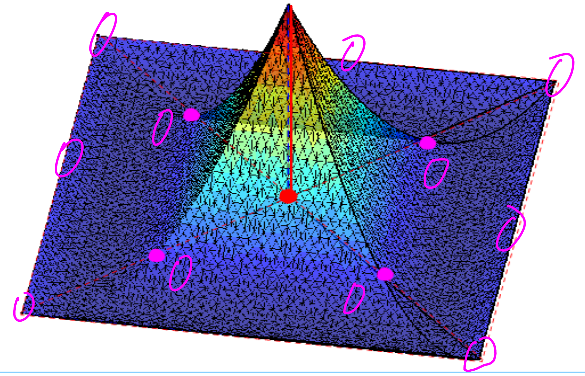
▷ $b_h^i|_e$ fixed

▷ continuity

continuous!

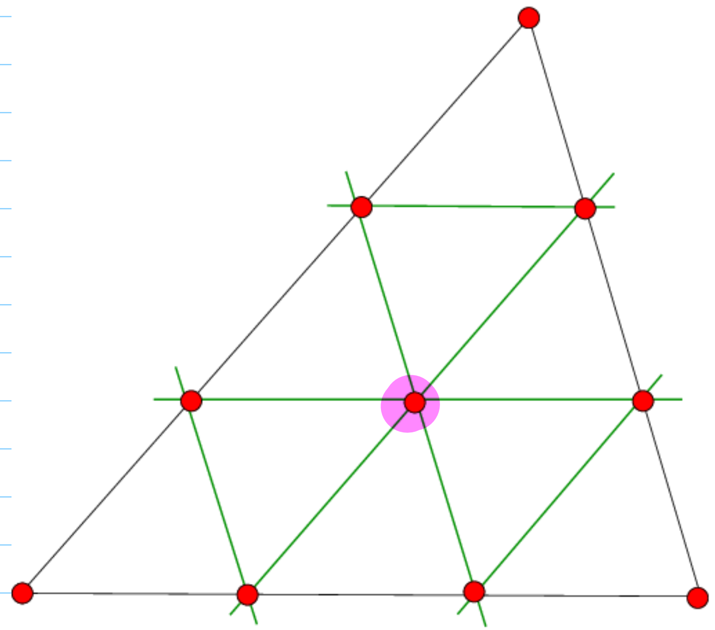




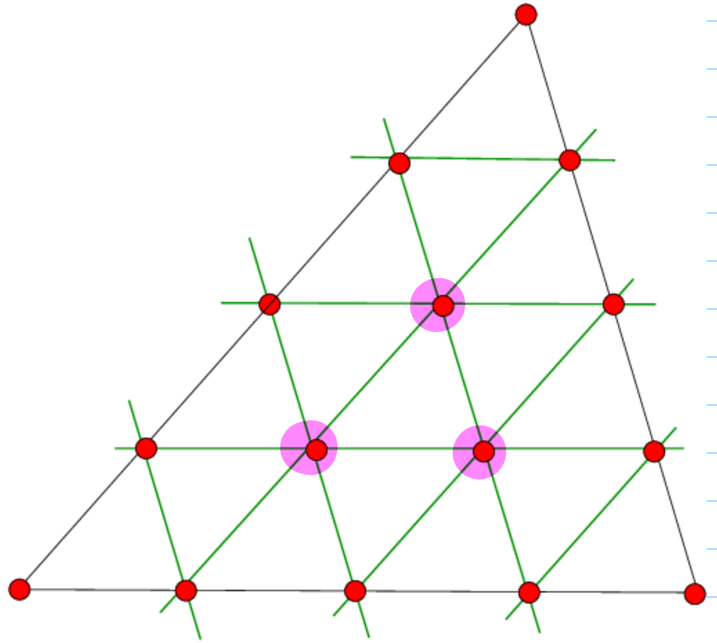


◁ Global basis function for $\mathcal{S}_2^0(\mathcal{M})$ associated with a vertex

③



(local) interpolation nodes for $\mathcal{S}_3^0(\mathcal{M})$

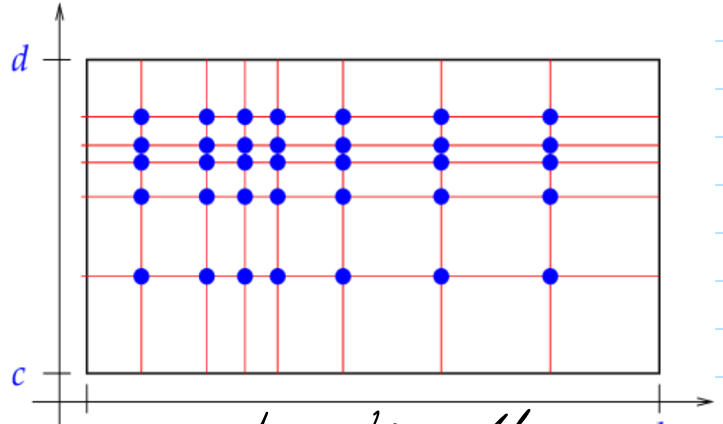


(local) interpolation nodes for $\mathcal{S}_4^0(\mathcal{M})$

2.6.2 Tensor-product Lagrange FEM

Another special type are tensor product meshes, also called **grids** ▷

in 2D: $a = x_0 < x_1 < \ldots < x_n = b\,,$
$c = y_0 < y_1 < \ldots < y_m = d\,.$



rectangular cells

▶ $\mathcal{M} = \{]x_{i-1}, x_i[\times]y_{j-1}, y_j[:\ 1 \le i \le n, 1 \le j \le m \}\,.$ (2.5.1.4)

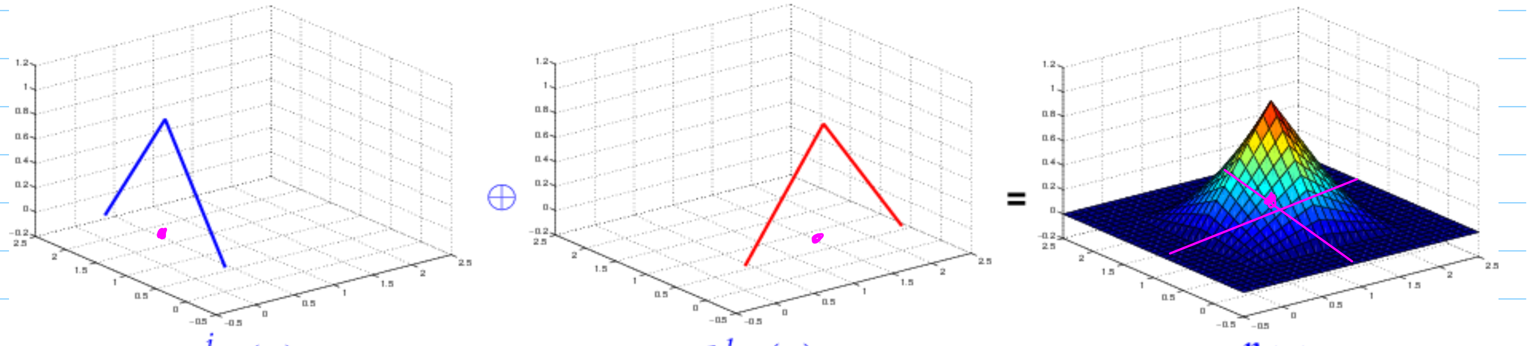
☞ Restricted to tensor product domains

Now use $\mathcal{M}$-p.w. tensor-product polynomials

$d=2$, $p=1$: Bilinear Lagrange FEM

↳ locally $\in \mathcal{Q}_1(\mathbb{R}^2)$

Interpolation nodes = nodes of $\mathcal{M}$



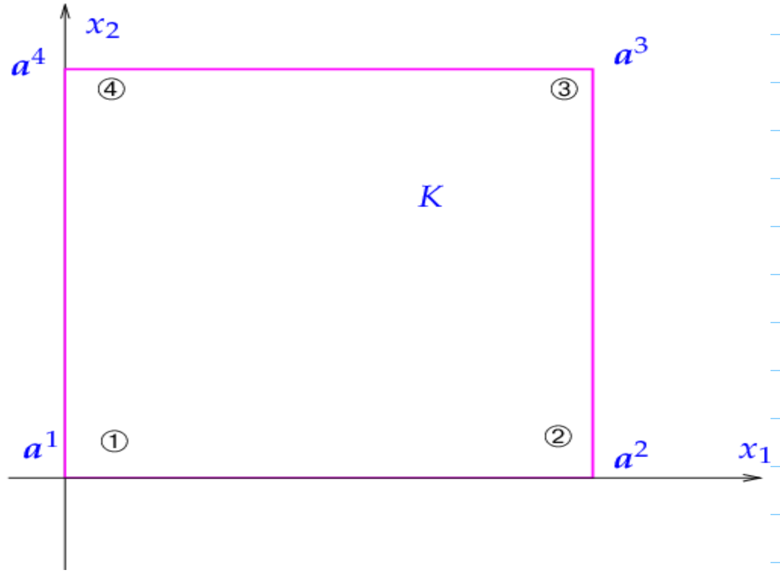
$b^j_{h,x}(x)$ ⊕ $b^l_{h,y}(y)$ = $b^p_h(x)$

$b_h^p\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = b^j_{h,x}(x)\, b^l_{h,y}(y)$

Local shape functions on $K =]0,1[^2$

$$\begin{aligned} b^1_K(\mathbf{x}) &= (1-x_1)(1-x_2)\,, \\ b^2_K(\mathbf{x}) &= x_1(1-x_2)\,, \\ b^3_K(\mathbf{x}) &= x_1 x_2\,, \\ b^4_K(\mathbf{x}) &= (1-x_1)x_2\,. \end{aligned} \quad \in \mathcal{Q}_1 \qquad (2.6.2.3)$$

▶ $b^i_K(\mathbf{a}^j) = \delta_{ij}\,,\quad 1 \le i,j \le 4\,,$



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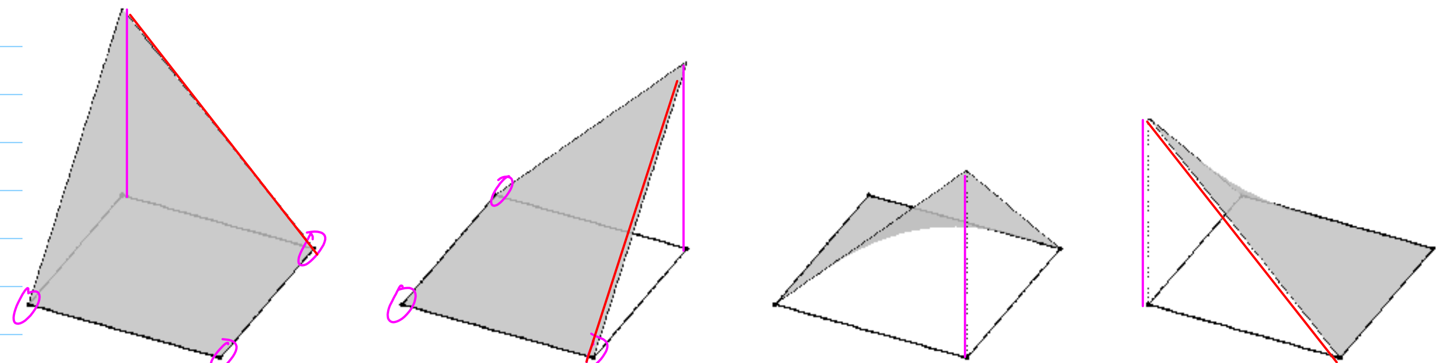


Bilinear local shape functions on unit square K

C^0-gluing possible because b_K^i linear on every edge

Definition 2.6.2.5. Tensor product Lagrangian finite element spaces

Space of p-th degree Lagrangian finite element functions on tensor product mesh $\mathcal{M}$

$$\mathcal{S}_p^0(\mathcal{M}) := \{v \in C^0(\overline{\Omega}) : v_{|K} \in \mathcal{Q}_p(K)\ \forall K \in \mathcal{M}\} .$$

↳ confirmed for $p = 1$

$p > 1$: Construct basis functions from cardinal basis property w.r.t. suitable interpolation nodes

EXAMPLE 2.6.2.7 (Quadratic tensor product Lagrangian finite elements) Consider the case $p = 2, d = 2$ for Def. 2.6.2.5:

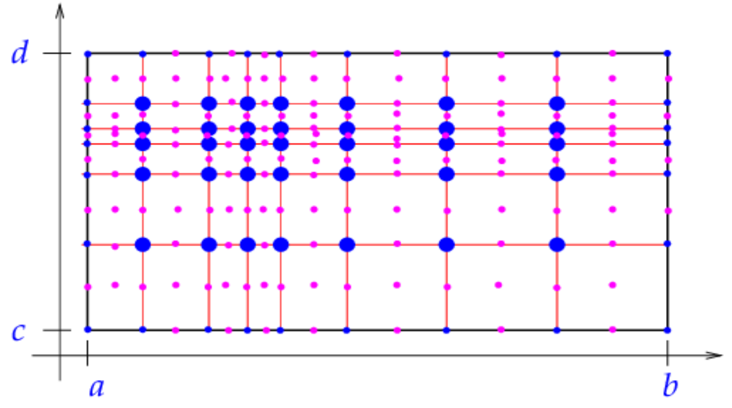
Interpolation nodes for $\mathcal{S}_2^0(\mathcal{M})$

$$\mathcal{N} = \mathcal{V}(\mathcal{M}) \cup \{\text{midpoints of edges}\} .$$

$\cup$ {center points of cells}

Note: number of interpolation nodes belonging to one cell is

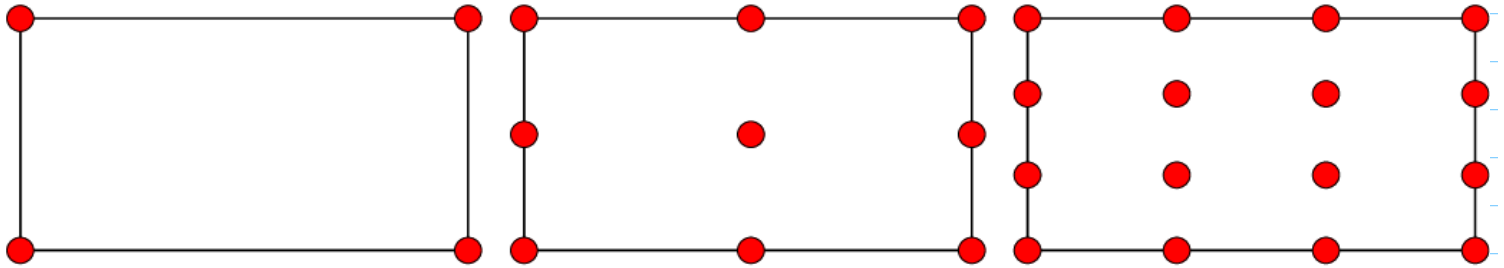
$$9 = \dim \mathcal{Q}_2(\mathbb{R}^2) .$$



$$\mathcal{N} = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_N\}: \quad b_h^j \in \mathcal{S}_2^0(\mathcal{M}), \quad b_h^j(\boldsymbol{p}_i) := \begin{cases} 1 & \text{, if } j = i , \\ 0 & \text{else.} \end{cases}$$

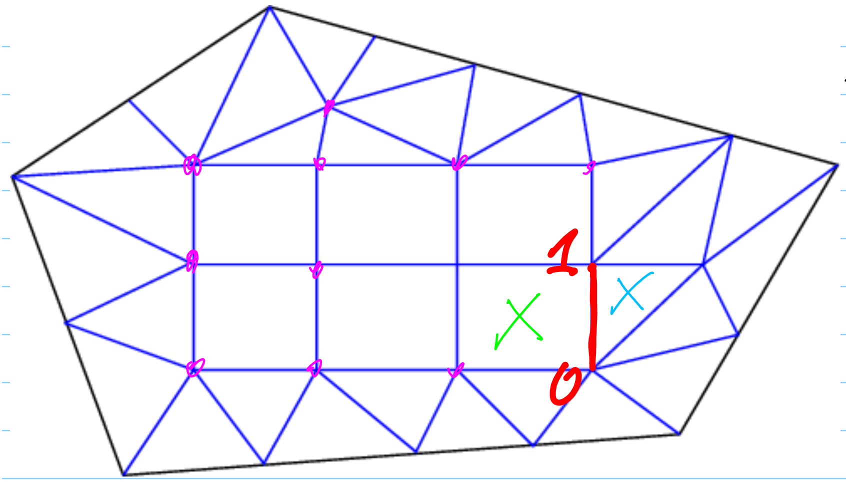
General pattern:

Choice of interpolation nodes for tensor product Lagrangian finite elements of higher degree:



$p = 1$ $\qquad$ $p = 2$ $\qquad$ $p = 3$

5 Extension: $(\mathcal{B}_1)$-linear Lagrangian FEM on hybrid mesh



▷ $\mathcal{M}$ has triangular & rectangular cells

Idea:

Use: $\mathcal{P}_1$ on triangles
$\mathcal{Q}_1$ on rectangles

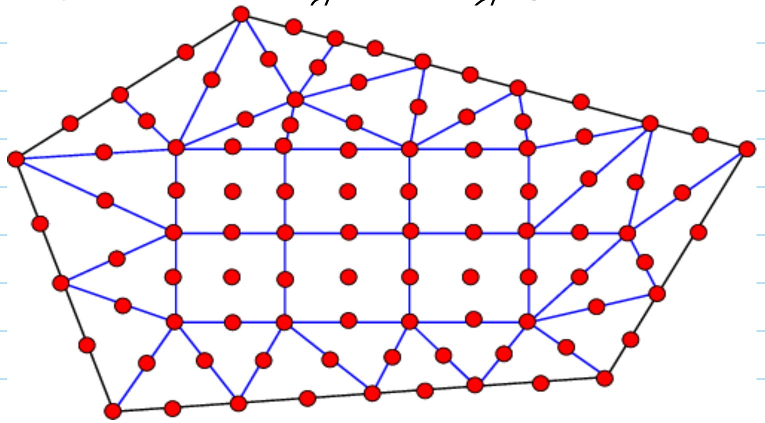
$$\mathcal{S}_1^0(\mathcal{M}) = \left\{ v \in H^1(\Omega) \colon v_{|K} \in \begin{cases} \mathcal{P}_1(\mathbb{R}^2) & \text{, if } K \in \mathcal{M} \text{ is triangle,} \\ \mathcal{Q}_1(\mathbb{R}^2) & \text{, if } K \in \mathcal{M} \text{ is rectangle} \end{cases} \right\}. \tag{2.6.2.11}$$

$[v \in C^0(\bar{\Omega})]$

Interpolation nodes $\hat{=}$ nodes of $\mathcal{M}$

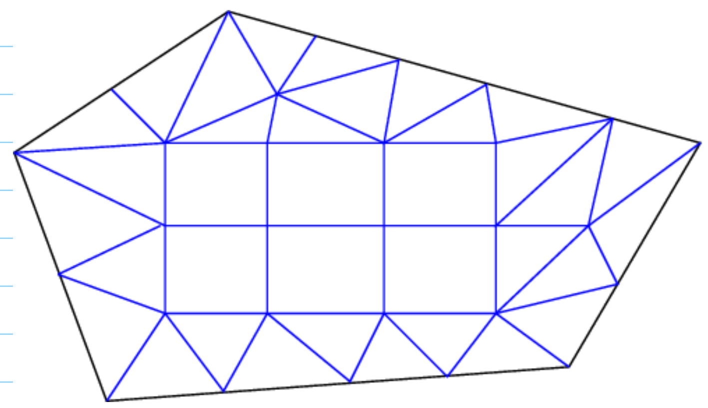
Gluing works, because functions $\in \mathcal{Q}_1/\mathcal{P}_1$ are all linear, when restricted to a straight edge

$p=2$: interpolation nodes ▷



6 Review questions:

A:



Explain why the local shape functions

- for $\mathcal{S}_1^0(\mathcal{M})$ on a triangular mesh (= the barycentric coordinate function of Ex. 2.5.3.6)
- and those for $\mathcal{S}_1^0(\mathcal{M})$ on a tensor-product mesh as defined by (2.6.2.3)

remain valid local shape functions for the lowest degree Lagrangian finite element space on the hybrid mesh displayed beside.

B:

Let $\mathcal{M}$ be a triangular mesh with $\sharp\mathcal{V}(\mathcal{M})$ vertices, $\sharp\mathcal{E}(\mathcal{M})$ edges, and $\sharp\mathcal{M}$ cells. What is $\dim \mathcal{S}_p^0(\mathcal{M})$ for $p = 1, 2, 3$?

C:

For a triangular mesh $\mathcal{M}$ with $\sharp\mathcal{V}(\mathcal{M})$ vertices, $\sharp\mathcal{E}(\mathcal{M})$ edges, and $\sharp\mathcal{M}$ cells give sharp upper bounds for the number of non-zero entries of the Galerkin matrix arising from the finite element discretization of

$$u \in H^1(\Omega): \quad \int_\Omega \alpha(\mathbf{x}) \,\mathbf{grad}\, u \cdot \mathbf{grad}\, v + c(\mathbf{x}) u\, v \,\mathrm{d}\mathbf{x} = \int_\Omega f v \,\mathrm{d}\mathbf{x} \quad \forall v \in H^1(\Omega)\,, \tag{2.1.2.2}$$

with trial and test space $\mathcal{S}_p^0(\mathcal{M})$, $p = 1, 2$.

D:

Consider a tensor product mesh $\mathcal{M}$ of $\Omega :=]0,1[^2$ with $n \in \mathbb{N}$ cells in each direction and the finite-element space

$$V_{0,h} := \left\{ v \in H_0^1(\Omega) \colon v_{|K} \in \mathcal{Q}_1(\mathbb{R}^2)\ \forall K \in \mathcal{M} \right\}.$$

What is the dimension of this space?

E:

Let $\mathcal{M}$ be a tensor product mesh of $\Omega :=]0,1[^2$ and $\widetilde{\mathcal{M}}$ a triangular mesh arising from $\mathcal{M}$ by splitting each rectangular cell into two congruent triangles. Show that $\mathcal{S}_1^0(\mathcal{M}) \neq \mathcal{S}_1^0(\widetilde{\mathcal{M}})$.

F:

Express the local shape functions for linear Lagrangian finite elements on a triangle as linear combinations of the quadratic local shape functions as given in

$$\begin{aligned} b_K^1 &= (2\lambda_1 - 1)\lambda_1\,, & b_K^2 &= (2\lambda_2 - 1)\lambda_2\,, \\ b_K^3 &= (2\lambda_3 - 1)\lambda_3\,, & b_K^4 &= 4\lambda_1\lambda_2\,, \\ b_K^5 &= 4\lambda_2\lambda_3\,, & b_K^6 &= 4\lambda_1\lambda_3\,. \end{aligned} \tag{2.6.1.6}$$

G:

Characterize the space of gradients of $\mathcal{P}_p(\mathbb{R}^2)$ and $\mathcal{Q}_p(\mathbb{R}^2)$ as spaces of componentwise polynomial vectorfields.

7

H:

We consider the bilinear form

$$\mathsf{b}(u,v) := \int_{\partial\Omega} (1+\|\boldsymbol{x}\|)\, u(\boldsymbol{x})v(\boldsymbol{x})\, \mathrm{d}S\,, \tag{2.6.2.14}$$

where $\partial\Omega$ is the boundary of the domain sketched in Fig. 120.

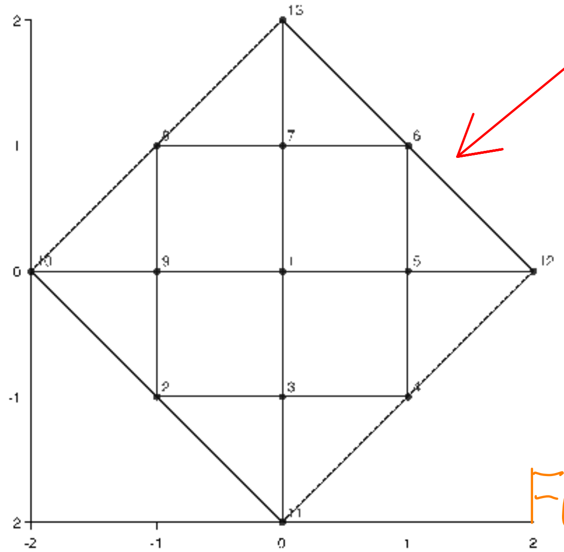


Fig 120

Writing $\mathcal{M}$ for the mesh drawn in Fig. 120, what is the maximal number of nonzero entries of the Galerkin matrix arising from the finite element Galerkin discretization of $\mathsf{b}(\cdot,\cdot)$ using $\mathcal{S}_1^0(\mathcal{M})$ equipped with the standard nodal basis as trial and test space?

I:

For the hybrid mesh $\mathcal{M}$ displayed in Fig. 120 determine the dimensions of the finite element spaces

(i) $\mathcal{S}_1^0(\mathcal{M}) := \left\{ v \in C^0(\overline{\Omega}) : \begin{array}{ll} v|_K \in \mathcal{P}_1(K) & \forall \text{triangles } K \in \mathcal{M} \\ v|_K \in \mathcal{Q}_1(K) & \forall \text{rectangles } K \in \mathcal{M} \end{array} \right\}$,

(ii) $\mathcal{S}_{1,0}^0(\mathcal{M}) := \mathcal{S}_1^0(\mathcal{M}) \cap H_0^1(\Omega)$.

J:

We perform the Galerkin discretization of

$$\mathsf{a}(u,v) := \int_{\partial\Omega} u(\boldsymbol{x})v(\boldsymbol{x})\, \mathrm{d}S\,,$$

where $\partial\Omega$ is the boundary of the domain from Fig. 120, by means of the finite element space $\mathcal{S}_1^0(\mathcal{M})$ equipped with the standard nodal basis. Here $\mathcal{M}$ is the hybrid mesh sketched in Fig. 120.
Compute the element matrices for the two cells of the mesh of Fig. 120 whose vertices are the nodes with the following numbers:

(i) nodes 9, 1, 7, 8 (quadrilateral),
(ii) nodes 3, 11, 4 (triangle)

Any local numbering of the nodes can be used.