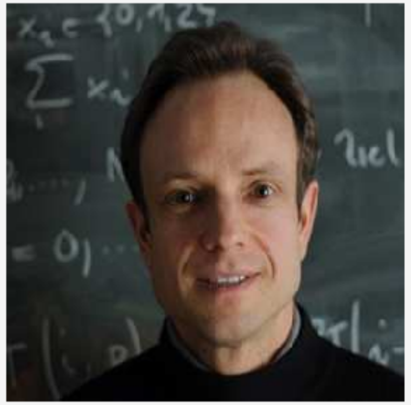
1

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 3.2: Empirical (Asymptotic) Convergence of FEM



Prof. R. Hiptmair, SAM, ETH Zurich

Date: March 16, 2019

Prerequisites.

- A clear idea about function spaces and associated norms
- Landau-O notation

Dependency. Lagrangian finite element methods [Lecture → Section 2.6], norms in Sobolev spaces [Lecture → Section 1.3], and numerical quadrature [Lecture → Section 2.7.5.2].

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

III. FEM: Convergence & Accuracy

3.2. Empirical (Asymptotic) Convergence of Lagrangian FEM

3.2.1. Asymptotic Convergence

What you have seen: asymptotic perspective

- quadrature errors ~ no. P of quad. pts → ∞
- polynomial intp. error ~ degree p → ∞
- [numerical integration ~ timestep τ → 0]

error studied as a function of a discretization parameter

Convergence: asymptotic perspective

Crucial: our notion of convergence is *asymptotic* !

sequence of discrete models ⇒ sequence of approximate solutions $(u_h^{(i)})_{i\in\mathbb{N}}$

⇒ study sequence $(\|u_h^{(i)} - u\|)_{i\in\mathbb{N}}$ as $i \to \infty$

created by *variation* of a discretization parameter.

② § 3.2.1.3:

For FEM: discretization parameter (DP)

- h-ref.: ▷ DP ≙ meshwidth

Definition 3.2.1.4. Mesh width

Given a mesh $\mathcal{M} = \{K\}$, its mesh width $h_{\mathcal{M}}$ is defined as

$$h_{\mathcal{M}} := \max\{\operatorname{diam} K : K \in \mathcal{M}\} \quad , \quad \operatorname{diam} K := \max\{|\mathbf{p} - \mathbf{q}| : \mathbf{p}, \mathbf{q} \in K\} .$$

Limit $h_{\mathcal{M}} \longrightarrow 0$

- p-ref.: DP $\hat{=}$ degree p

"Generic" Galerkin DP: $N := \dim V_{0,h}$
with limit $N \to \infty$ ↳ no. of dofs./unknowns

Relevant: $N \sim$ "cost of a FEM"

[Alternative DP: nnz(A)]

3.2.2 Algebraic & exponential Convergence

→ typical asymptotic convergence behavior

Definition 3.2.2.1. Types of convergence → ,

$\|u - u_N\| = O(N^{-\alpha}), \alpha > 0$:⟺ **algebraic convergence** with rate α

$\|u - u_N\| = O(\exp(-\gamma N^{\delta}))$, with $\gamma, \delta > 0$:⟺ **exponential convergence**

(asymptotically for $N \to \infty$)

rate of alg cvg.

Rem 3.2.2.4: Sharpness implied!

§ 3.2.2.5: (Exploring convergence empirically)

Detecting alg. cvg. from data (N_i, ε_i)

$\varepsilon_i \approx C N_i^{-\alpha}$, $C > 0$ unknown ↑ error norms

$\log \varepsilon_i \approx \log C - \alpha \log N_i$

[⇒ ≈ straight line in doubly logarithmic plot,
slope ⟷ α
↳ determine by linear regression]

③ 3.2.3. Convergence of FEM: Numerical Experiments

Model problem: $-\Delta u = f$ in Ω
$\Downarrow$ $\quad u = g$ on $\partial\Omega$
$\|\cdot\|_a = |\cdot|_{H^1(\Omega)}$

Below: Manufactured solutions: u given by a simple analytic expression $\Rightarrow f, g = u_{|\partial\Omega}$

Remark 3.2.3.3:

(Integral) error norms $\|u-u_h\|_{L^2(\Omega)}$, $|u-u_h|_{H^1(\Omega)}$, $\|u-u_h\|_\infty$ can usually be computed only by numerical quadrature (sampling) of sufficiently high order:

quad. error $\ll \|u-u_h\|_a$

Exp 3.2.3.6: 1D, $S_1^0(\mathcal{M})$

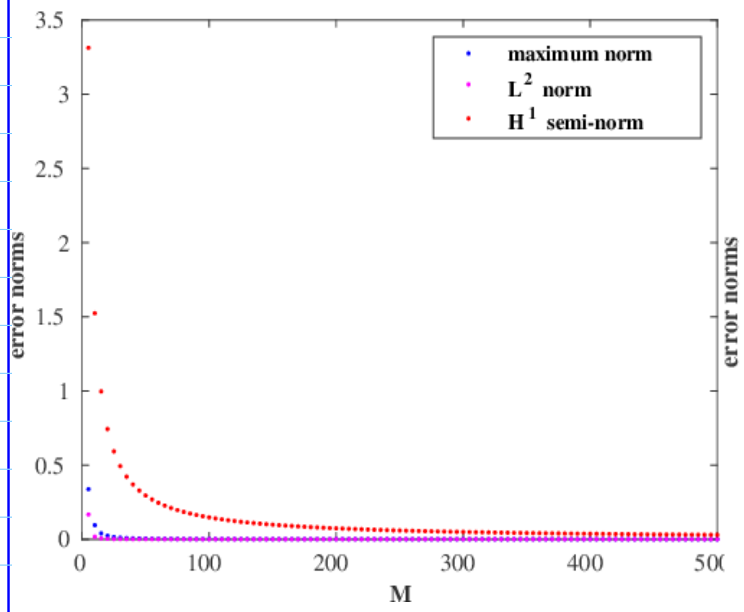
- domain $\Omega =]0,1[$,
- ODE: $-\dfrac{d^2u}{dx^2} = g$ in Ω,
- load $g(x) = -4\pi(\cos(2\pi x^2) - 4\pi x^2 \sin(2\pi x^2))$,
- boundary values $u_a = u_b = 0$.

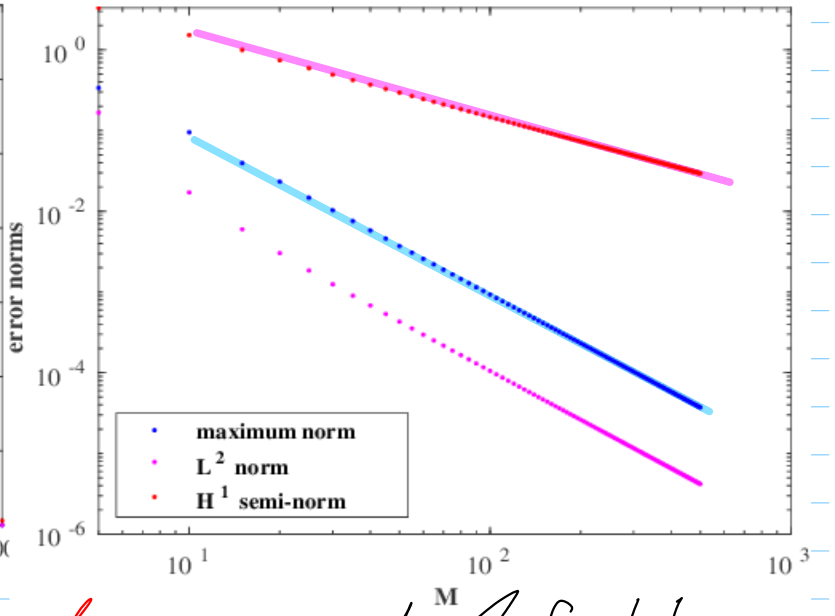
$\blacktriangleright$ unique solution

$u(x) = \sin(2\pi x^2)$. $\quad 0 < x < 1$.

("manufactured solution")

· equidistant mesh, M cells
· norm by 4th-order QR





$\triangleright$ alg. cvg.: rate 1 for $|\cdot|_{H^1}$, rate 2 for $\|\cdot\|_0$, $\|\cdot\|_\infty$

(4) Exp 3.2.3.7 : 2D , $\mathcal{S}_p^0(\mathcal{M})$, $p = 1, 2$

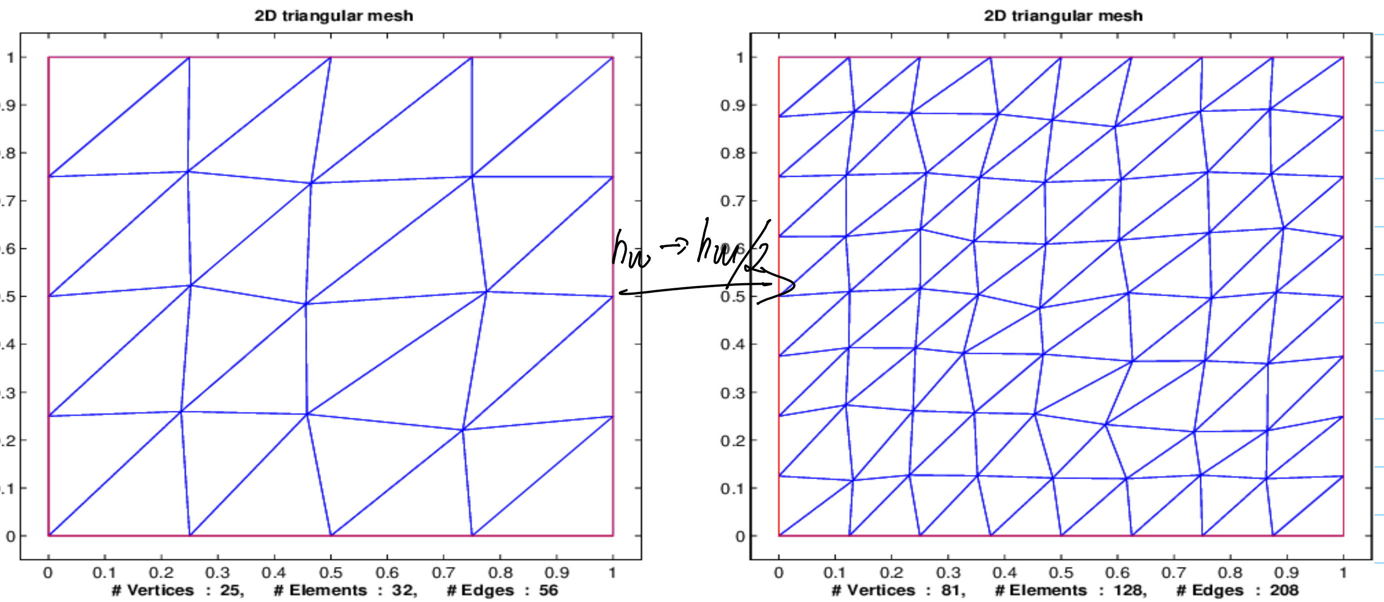
Setting: $\Omega =]0,1[^2$, $f(x_1, x_2) = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$, $\boldsymbol{x} \in \Omega$, $g = 0$

➢ Smooth solution $u(x,y) = \sin(\pi x)\sin(\pi y)$.

- Galerkin finite element discretization based on triangular meshes and
  - linear Lagrangian finite elements, $V_{0,N} = \mathcal{S}_{1,0}^0(\mathcal{M}) \subset H_0^1(\Omega)$ (→ Section 2.4),
  - quadratic Lagrangian finite elements, $V_{0,N} = \mathcal{S}_{2,0}^0(\mathcal{M}) \subset H_0^1(\Omega)$ (→ Ex. 2.6.1.2),
- quadrature rule (2.7.5.37) for assembly of local load vectors (→ Section 2.7.5),

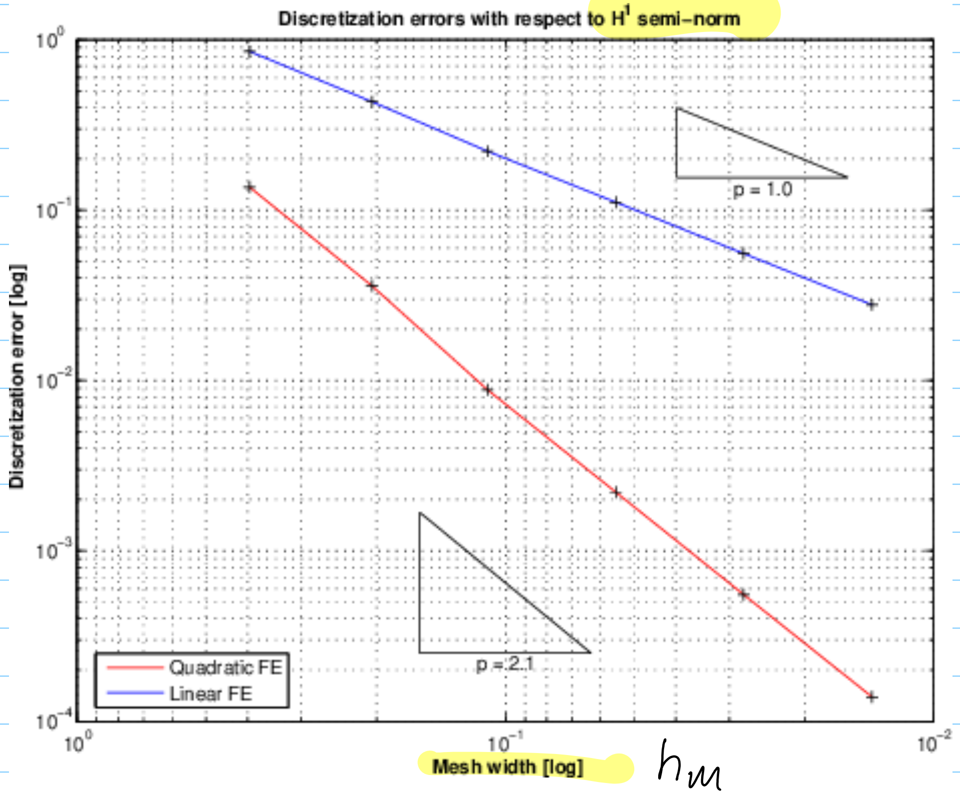
Monitored: $H^1(\Omega)$-semi-norm (→ Def. 1.3.4.3) of the Galerkin discretization error $u - u_N$

- sequences of meshes obtained by regular ref. (+ small deformation)



$h_{\mathcal{M}} \to h_{\mathcal{M}}/2$

Unstructured triangular meshes of $\Omega =]0,1[^2$ (two coarsest specimens)



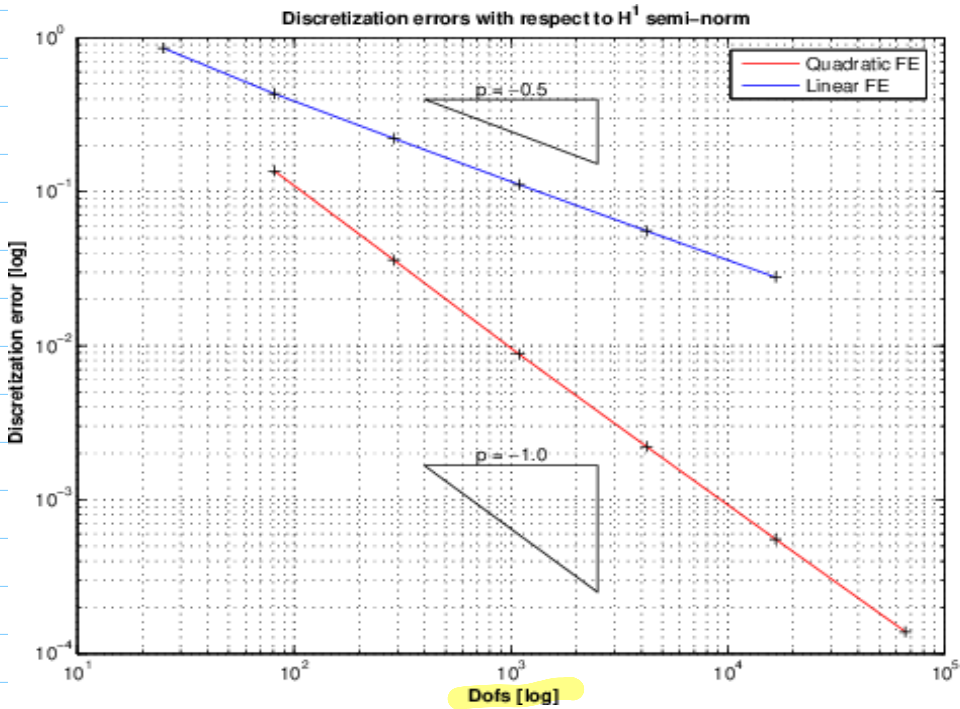
$h_{\mathcal{M}}$

⊲ log-log plot

→ alg. cvg.

$p = 1$: rate 1

$p = 2$: rate 2



→ alg. cvg. (w.r.t. $N = \dim \mathcal{S}_p^0(\mathcal{M})$)

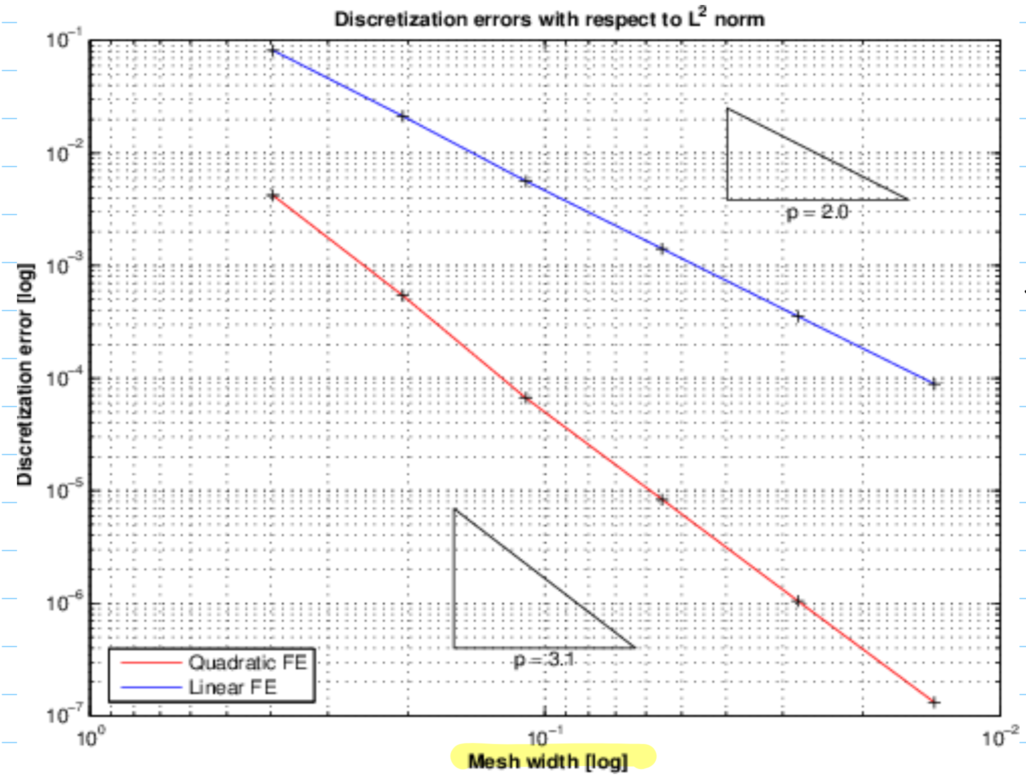
$p = 1$: rate $\frac{1}{2}$

$p = 2$: rate 1

Counting argument in 2D / regular refinement

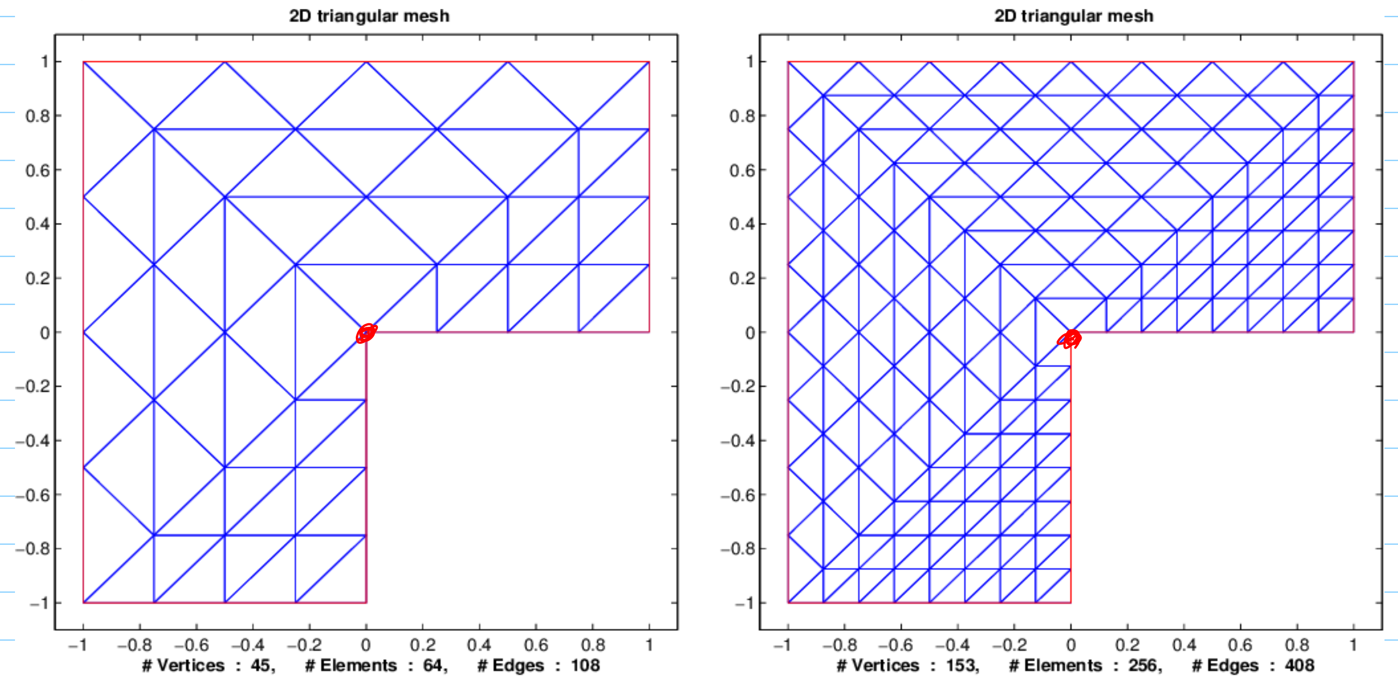
p-fixed: $N := \dim \mathcal{S}_p^0(\mathcal{M}) = O(h_{\mathcal{M}}^{-2})$ for $h_{\mathcal{M}} \to 0$

Exp 3.2.38: Same setting as before, focus on $\|u-u_N\|_{L^2(\Omega)}$



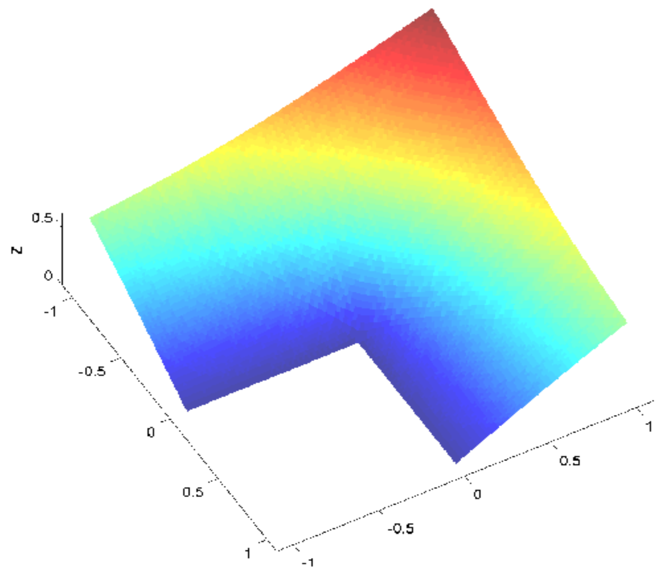
◁ alg. cvg. rates: $O(h_{\mathcal{M}}^{p+1})$

Exp 3.2.3.10: 2D, L-shaped domain



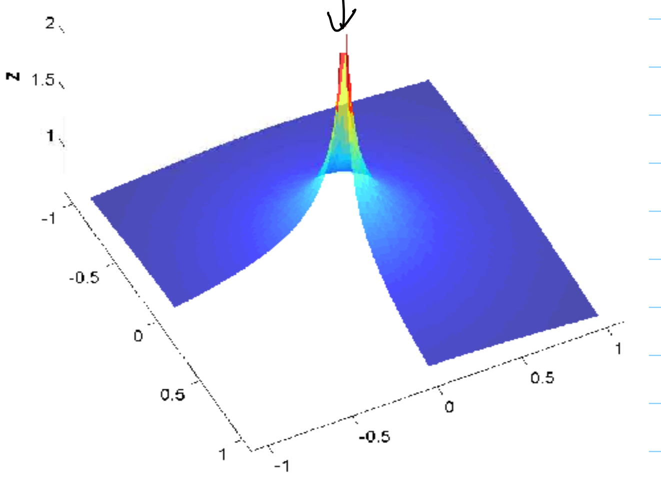
Unstructured triangular meshes of $\Omega =]-1,1[^2 \setminus (]0,1[\times]-1,0[)$ (two coarsest specimens)

Non-smooth solution: $u(r,\varphi) = r^{2/3}\sin(2/3\,\varphi)$ in polar coords



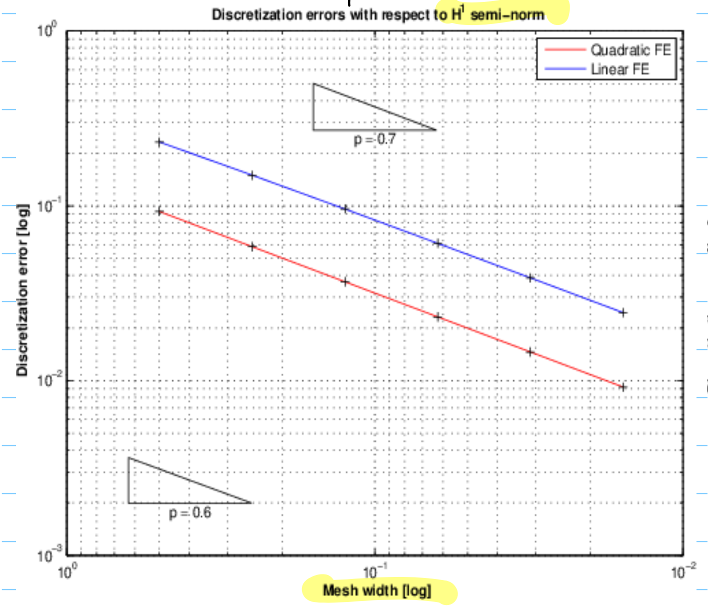
Exact solution u

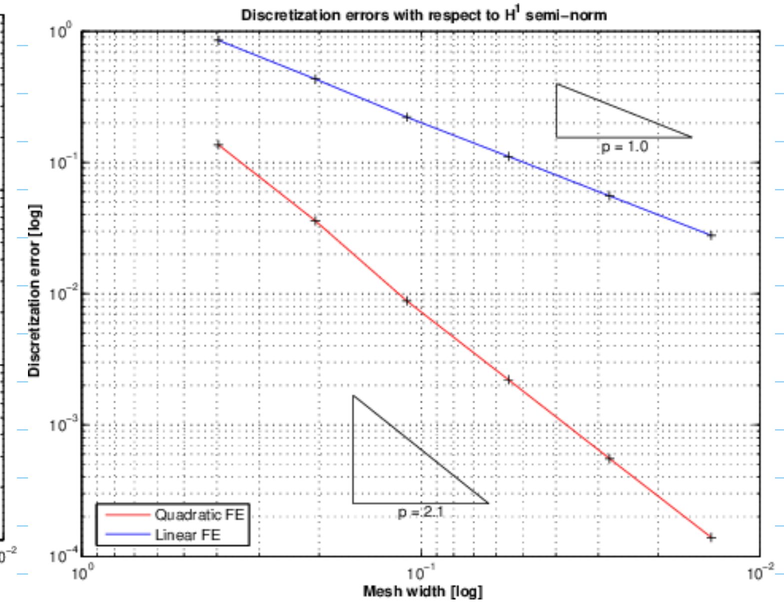
singularity at 0!



Norm of gradient: $\|\mathbf{grad}\, u\|$

⑥ L-shaped $\qquad \Omega = \square$

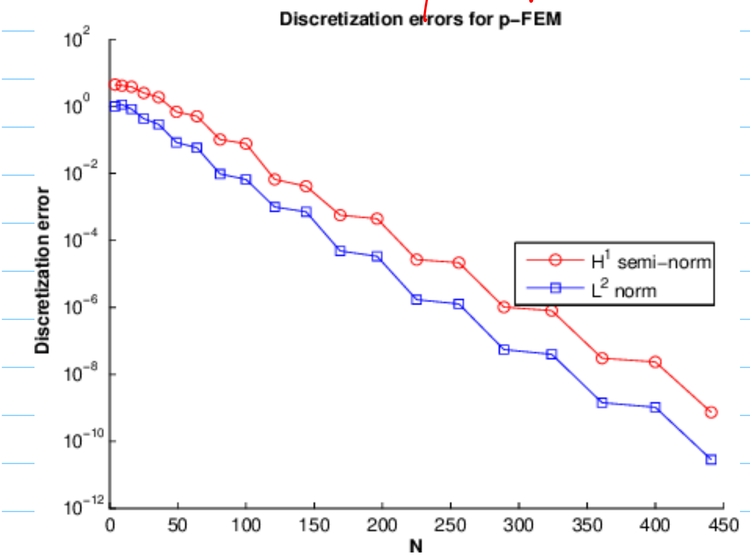




▷ alg. cvg: $O(h^{2/3})$

[same for $p=1,2$!]

Exp 3.2.3.11: 2D, $\Omega = \square$, smooth solution, p-refinement



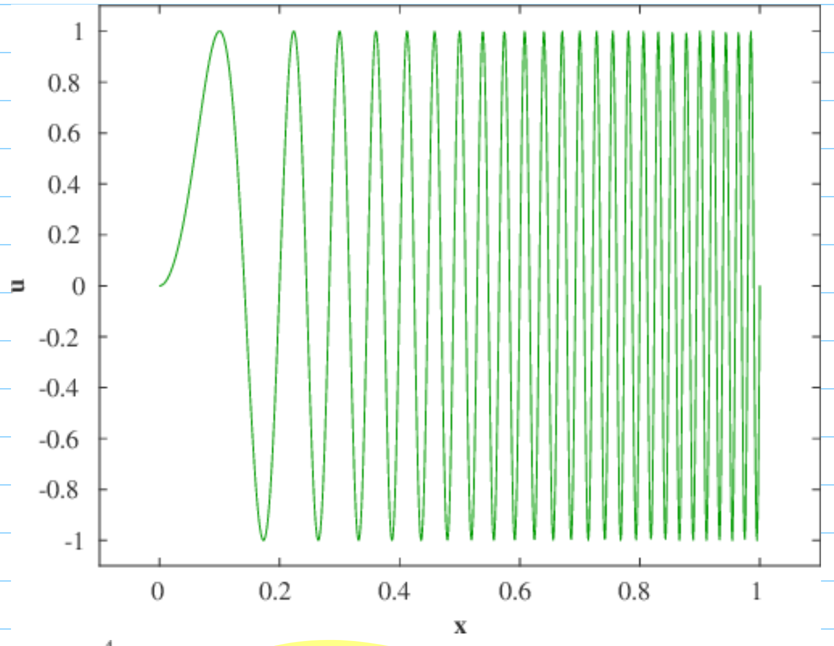
$$\log \varepsilon_i \approx C - \delta N_i$$

$$\varepsilon_i \approx C' e^{-\delta N_i}$$

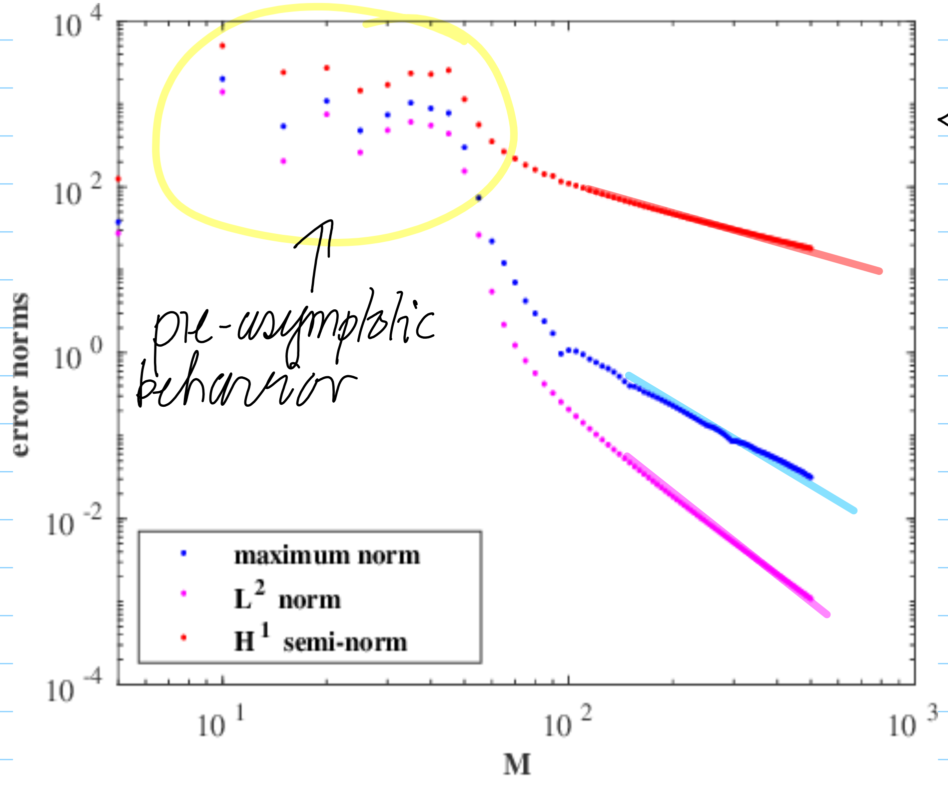
▷ exp. cvg.

[lin-log scale]

Exp 3.2.3.12: 1D, $S_1^0(\mathcal{M})$, equidistant meshes



◁ highly oscillatory C^∞-solution



◁ Delayed onset of alg. cvg.

▽ Convergence can be "really asymptotic"

§3.2.3.13: Summary:

- h-refinement $\Longrightarrow$ alg. cvg.
- L^2-cvg. faster than $\|\cdot\|_a$-cvg. ?
- Asymptotic convergence depends on smoothness of u !

Review questions 3.2.3.14 :

[Please tackle without resorting to other sources of information]

A :

You forget the call to `std::sqrt()` when computing approximations of $\|u-u_h\|_{L^2(\Omega)}$ and $|u-u_h|_{H^1(\Omega)}$, $u-u_h$ a finite-element Galerkin discretization error for a second-order elliptic BVP. What will be the impact on observed rates of algebraic convergence?

B :

In a numerical experiment we observe the following asymptotic convergence for the finite-element solutions of a second-order elliptic BVP on $\Omega \subset \mathbb{R}^2$ and h-refinement $(h_{\mathcal{M}} \to 0)$

$$|u-u_h|_{H^1(\Omega)} = O(h_{\mathcal{M}}) \quad , \quad \|u-u_h\|_{L^2(\Omega)} = O(h_{\mathcal{M}}^2) \; .$$

What asymptotic convergence for $h_{\mathcal{M}} \to 0$ do you predict for the error norm $\|u-u_h\|_{L^2(\partial\Omega)}$?

Hint. You can refer the the following result

Theorem 1.9.0.10. Multiplicative trace inequality

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \le C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) \; .$$

C :

The following table list error norms recorded for a sequence of finite element solutions $u_\ell \in H^1(\Omega)$ belonging to finite element spaces with dimensions N_ℓ. From the data predict qualitatively and quantitatively the asymptotic convergence for $N_\ell \to \infty$.

N_ℓ	8	16	321	64	128	256	512	1024
$\|u-u_\ell\|_{H^1(\Omega)}$	4.98e-01	4.08e-01	3.23e-01	2.57e-01	2.01e-01	1.53e-01	1.26e-01	1.02e-01
$\|u-u_\ell\|_{L^2(\Omega)}$	3.57e-01	2.55e-01	1.79e-01	1.24e-01	8.89e-02	6.15e-02	4.49e-02	3.05e-02

D :

We solve a second-order elliptic BVP based on the Lagrangian finite-element spaces $\mathcal{S}_1^0(\mathcal{M}_\ell)$ and $\mathcal{S}_2^0(\mathcal{M}_\ell)$, and on a sequence of triangular meshes $(\mathcal{M}_\ell)_{\ell=0}^L$ obtained by uniform regular refinement. We get the finite element Galerkin solutions $u_\ell^1 \in \mathcal{S}_1^0(\mathcal{M}_\ell)$ and $u_\ell^2 \in \mathcal{S}_2^0(\mathcal{M}_\ell)$, $\ell = 0,\dots,L$.
How will the presence of a singularity of $\mathbf{grad}\, u$, $u \in H^1(\Omega)$ the exact solution, manifest itself in the asymptotic behavior of the $L^2(\Omega)$- and $H^1(\Omega)$-norms of the finite element discretization errors?

E :

How should you read the following statement?

> "Exponentially convergent Galerkin schemes are better than algebraically convergent methods"

This list of review questions may not be complete. Additional review questions may be provided in the lecture document.