(1)

ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 3.3: A Priori (Asymptotic) Finite Element Error Estimates (II)

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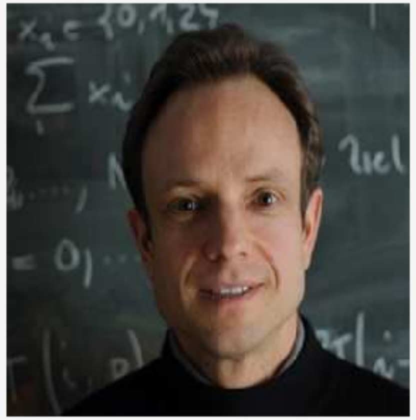


Prerequisites.

- Knowledge of polynomial interpolation in 1D and related error estimates

Dependency. [Lecture → Section 2.6] and [Lecture → Section 3.1]

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

III. FEM : Accuracy & Convergence

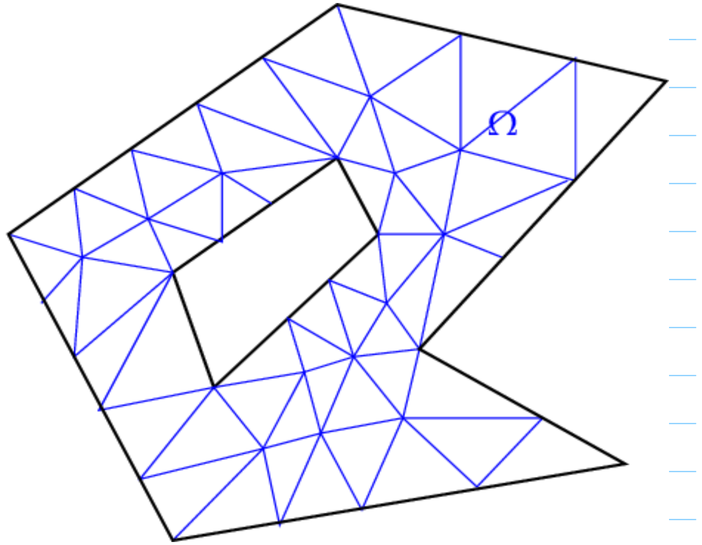
3.3. A Priori (Asymptotic) Finite Element Error Estimates

3.3.2. Error Estimates for Piecewise Linear Interpolation in 2D

Given:

- polygonal domain $\Omega \subset \mathbb{R}^2$
- triangular mesh $\mathcal{M}$ of Ω

(→ Def. 2.5.1.1)



Definition 3.3.2.1. Linear interpolation in 2D

The linear interpolation operator $\mathsf{I}_1 : C^0(\bar{\Omega}) \mapsto \mathcal{S}_1^0(\mathcal{M})$ is defined by

$$\mathsf{I}_1 u \in \mathcal{S}_1^0(\mathcal{M}) \quad , \quad \mathsf{I}_1 u(\boldsymbol{p}) = u(\boldsymbol{p}) \quad \forall \boldsymbol{p} \in \mathcal{V}(\mathcal{M}) \, .$$

(2)

By cardinal basis property:

$$I_1 u = \sum_{j=1}^{N} u(x_j)\, b_h^j \;, \; N := \#\mathcal{V}(\mathcal{M})$$

Lemma 3.3.2.18. Local interpolation error estimates for 2D linear interpolation

For any triangle K and $u \in C^2(\overline{K})$ the following holds → smoothness requirement

$$\|u - \mathsf{I}_1 u\|^2_{L^2(K)} \le \tfrac{3}{8} h_K^4 \left\| \left\| D^2 u \right\|_F \right\|^2_{L^2(K)} , \tag{3.3.2.13}$$

$$\|\mathbf{grad}\,(u - \mathsf{I}_1 u)\|^2_{L^2(K)} \le \frac{3}{24} \frac{h_K^6}{|K|^2} \left\| \left\| D^2 u \right\|_F \right\|^2_{L^2(K)} . \tag{3.3.2.17}$$

NEW: shape of K enter here

↳ $D^2 u \mathrel{\hat=}$ Hessian of u

$\|\cdot\|_F \mathrel{\hat=}$ Frobenius norm

Definition 3.3.2.20. Shape regularity measure

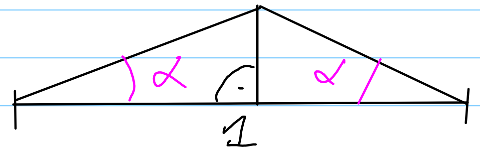
For a simplex $K \in \mathbb{R}^d$ we define its **shape regularity measure** as the ratio

$$\rho_K := h_K^d : |K| \,,$$

and the shape regularity measure of a simplicial mesh $\mathcal{M} = \{K\}$

$$\rho_{\mathcal{M}} := \max_{K \in \mathcal{M}} \rho_K \,.$$

▷ ρ_K is similarity-invariant

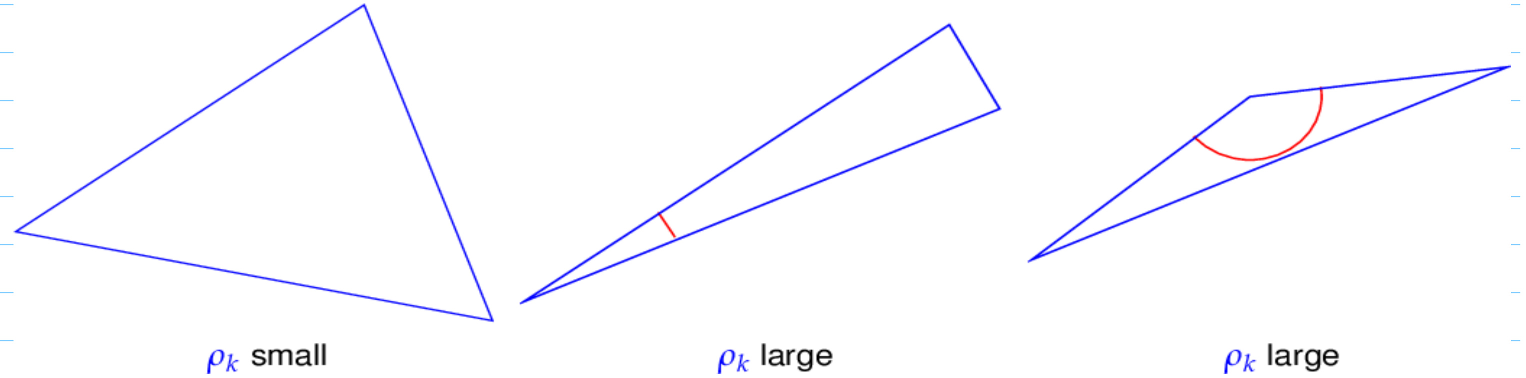


$|K| \geq \tfrac{1}{4} \tan\alpha$

$\rho_K \leq \tfrac{4}{\tan\alpha} \longrightarrow \infty$

as $\alpha \longrightarrow 0$

For triangle K: ρ_K large $\Leftrightarrow$ K "distorted" $\Leftrightarrow$ K has small angles



Theorem 3.3.2.21. Error estimate for piecewise linear interpolation

For any $u \in C^2(\bar{\Omega})$ and 2D piecewise linear interpolation $\mathsf{I}_1 : C^0(\overline{\Omega}) \to \mathcal{S}_1^0(\mathcal{M})$, $\mathcal{M}$ a triangular mesh, holds

$$\|u - \mathsf{I}_1 u\|_{L^2(\Omega)} \le \sqrt{\tfrac{3}{8}}\, h_{\mathcal{M}}^2 \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)} ,$$

$$\|\mathbf{grad}\,(u - \mathsf{I}_1 u)\|_{L^2(\Omega)} \le \sqrt{\tfrac{3}{24}}\, \rho_{\mathcal{M}} h_{\mathcal{M}} \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)} .$$

where $h_{\mathcal{M}}$ denotes the mesh width (→ Def. 3.2.1.4) and $\rho_{\mathcal{M}}$ the shape regularity measure (→ Def. 3.3.2.20) of $\mathcal{M}$.

$|\cdot|_{H^1}$: If $\rho_{\mathcal{M}}$ uniformly bounded then asympt. alg. cvg. of $\|u - I_1 u\|_x$ with rate 2 for L^2-norm, rate 1 for H^1-seminorm

3.3.3. The Sobolev Scale of Function Spaces

$d=2$:

$$\| \|D^2 u\|_F \|^2_{L^2(\Omega)} = \int_\Omega \sum_{j,i=1}^{2} \left| \frac{\partial^2 u}{\partial x_i \partial x_j}(x)\right|^2 dx$$

$$\Updownarrow$$

$$|u|^2_{H^1(\Omega)} = \int_\Omega \sum_{j=1}^{d} \left|\frac{\partial u}{\partial x_j}(x)\right|^2 dx$$

Definition 3.3.3.1. Higher order Sobolev spaces/norms

The m-th order Sobolev norm, $m \in \mathbb{N}_0$, for $u : \Omega \subset \mathbb{R}^d \mapsto \mathbb{R}$ (sufficiently smooth) is defined by

$$\|u\|^2_{H^m(\Omega)} := \sum_{k=0}^{m} \sum_{\alpha\in\mathbb{N}^d, |\alpha|=k} \int_\Omega |D^\alpha u|^2 \, dx\,, \quad \text{where} \quad D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}\,.$$

Sobolev space $\quad H^m(\Omega) := \{v : \Omega \mapsto \mathbb{R}: \|v\|_{H^m(\Omega)} < \infty\}$.

$\triangleright \quad L^2(\Omega) = H^0(\Omega) \supset H^1(\Omega) \supset H^2(\Omega) \supset H^3(\Omega) \supset \ldots$

$\longrightarrow$ increasing regularity of functions

$\|\cdot\|_{H^m}$ measures "ruggedness" of functions

Definition 3.3.3.3. Higher order Sobolev semi-norms

The m-th order Sobolev semi-norm, $m \in \mathbb{N}$, for sufficiently smooth $u : \Omega \mapsto \mathbb{R}$ is defined by

$$|u|^2_{H^m(\Omega)} := \sum_{\alpha\in\mathbb{N}^d, |\alpha|=m} \int_\Omega |D^\alpha u|^2 \, dx\,.$$

Corollary 3.3.3.4. Error estimate for piecewise linear interpolation in 2D

Under the assumptions/with notations of Thm. 3.3.2.21

$$\begin{aligned}\|u - \mathrm{I}_1 u\|_{L^2(\Omega)} &\le \sqrt{\tfrac{3}{8}}\, h^2_{\mathcal{M}} |u|_{H^2(\Omega)}\,, \\ |u - \mathrm{I}_1 u|_{H^1(\Omega)} &\le \sqrt{\tfrac{3}{24}}\, \rho_{\mathcal{M}}\, h_{\mathcal{M}} |u|_{H^2(\Omega)}\,,\end{aligned} \qquad \forall u \in H^2(\Omega)\,.$$

When $|u|_{H^m} = 0$?

$\Leftrightarrow \quad u \in \mathcal{P}_{m-1}(\mathbb{R}^d)$

$u \in H^m(\Omega)$ also provides information about the continuity of u:

C depends on Ω

Theorem 3.3.3.8. Sobolev embedding theorem

$$m > \frac{d}{2} \Rightarrow H^m(\Omega) \subset C^0(\overline{\Omega}) \wedge \exists C = C(\Omega) > 0: \; \|u\|_\infty \le C\|u\|_{H^m(\Omega)} \quad \forall u \in H^m(\Omega)\,.$$

$d=2$, $d=3$ $\Rightarrow m \ge 2$: functions in $H^2(\Omega)$ are continuous up to $\partial\Omega$

Why is an estimate like

$$\|u - I_1 u\|_{L^2(\Omega)} \le C h_{\mathcal{M}} |u|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)$$

not possible ?

(4) Review questions [c R.Q 3.3.5.29]

[Answer honestly without aids!]

A:

What is meant by the following statement?

> The nodal interpolation operator onto $\mathcal{S}_2^0(\mathcal{M})$, $\mathcal{M}$ a triangular mesh of some polygonal domain Ω, is *purely local*.

B:

If $\mathcal{M}'$ has been created by regular refinement of a triangular mesh $\mathcal{M}$, how are shape-regularity measures $\rho_{\mathcal{M}}$ and $\rho_{mesh'}$ related?

Definition 3.3.2.20. Shape regularity measure

For a simplex $K \in \mathbb{R}^d$ we define its **shape regularity measure** as the ratio

$$\rho_K := h_K^d : |K| \, ,] \quad h_K := \operatorname{diam}(K) \, ,$$

and the shape regularity measure of a simplicial mesh $\mathcal{M} = \{K\}$ as

$$\rho_{\mathcal{M}} := \max_{K \in \mathcal{M}} \rho_K \, .$$

C:

For which exponents $a > 0$ does the function $x \mapsto x^\alpha$ belong to the Sobolev space $H^m(]0,1[)$, $m \in \mathbb{N}$?

Hint. Since the function belongs to $C^\infty(]0,1[)$, it is sufficient to show that the improper integral defining its $H^m(]0,1[)$-norm has a finite value.

D:

Characterize the set of functions

$$Z := \{v \in H^p(\Omega) : \; |v|_{H^p(\Omega)} = 0\} \, .$$

Discuss the implications of your insights for the following theorem.

Theorem 3.3.3.4. Error estimate for piecewise linear interpolation in 2D

Under the assumptions/with notations of Thm. 3.3.2.21

$$\begin{aligned} \|u - \mathsf{I}_1 u\|_{L^2(\Omega)} &\leq \sqrt{\tfrac{3}{8}}\, h_{\mathcal{M}}^2 |u|_{H^2(\Omega)} \, , \\ |u - \mathsf{I}_1 u|_{H^1(\Omega)} &\leq \sqrt{\tfrac{3}{24}}\, \rho_{\mathcal{M}}\, h_{\mathcal{M}} |u|_{H^2(\Omega)} \, , \end{aligned} \qquad \forall u \in H^2(\Omega) \, .$$

⚠ This list of review questions may not be complete. Additional review questions may be provided in the lecture document.

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