ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 3.5: Variational Crimes

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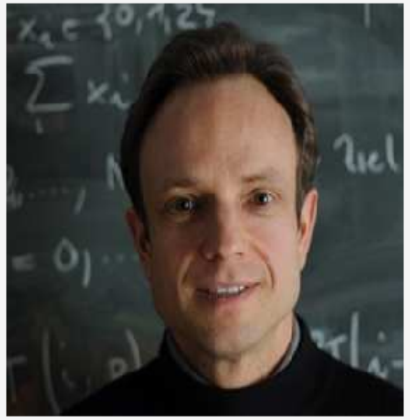


Dependency. [Lecture → Section 3.3] and [Lecture → Section 2.7.5.2]

[Slightly updated version of tablet notes]

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

III. FEM: Convergence & Accuracy

3.5. Variational Crimes

Variational crime = instead of solving (exact) discrete (linear) variational problem

$$u_h \in V_{0,h}: \quad a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_{0,h}, \tag{2.2.1.1}$$

we solve the perturbed variational problem (*)

$$\tilde{u}_h \in V_{0,h}: \quad a_h(\tilde{u}_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_{0,h}. \tag{3.5.0.1}$$

this causes a perturbation of Galerkin solution u_h and we end up with a perturbed solution $\tilde{u}_h \in V_{0,h}$.

(*) due to – the use of numerical quadrature
– boundary approximations

Which perturbations are acceptable.

Guideline for acceptable variational crimes

Variational crimes must not affect (type and rate) of asymptotic convergence!

$\|u - \tilde{u}_h\| \to 0$ as fast as $\|u - u_h\| \to 0$

[for, e.g., $h_M \to 0$]

(2)

5.5.1. Impact of Numerical Quadrature

Model problem: on polygonal/polyhedral $\Omega \subset \mathbb{R}^d$:

$$u \in H_0^1(\Omega): \quad \mathsf{a}(u,v) := \int_\Omega \sigma(\mathbf{x})\, \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}\mathbf{x} = \ell(v) := \int_\Omega f v \,\mathrm{d}\mathbf{x} \,. \qquad (3.5.1.1)$$

Assumptions: σ satisfies (1.6.0.6), $\sigma \in C^0(\overline{\Omega})$, $f \in C^0(\overline{\Omega})$)

- Galerkin finite element discretization, $V_h := \mathcal{S}_p^0(\mathcal{M})$ on simplicial mesh $\mathcal{M}$
- Approximate evaluation of $\mathsf{a}(u_h, v_h)$, $\ell(v_h)$ by a fixed stable local numerical quadrature rule (→ Section 2.7.5) $\rightarrow a_h, \ell_h$

➤ perturbed bilinear form a_h, right hand side ℓ_h (see (3.5.0.1))

Experiment 3.5.1.2 : (h-refinement)

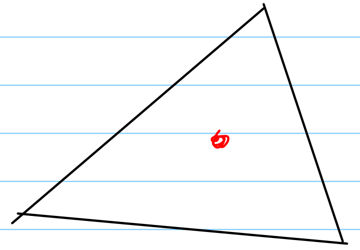
→ smooth solution $\in H^k(\Omega)$

$\Omega =]0,1[^2$, $\sigma \equiv 1$, $f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$, $(x,y)^T \in \Omega$

➤ solution $u(x,y) = \sin(\pi x)\sin(\pi y)$, $g = 0$.

Details of numerical experiment:

- Quadratic Lagrangian FE ($V_h = \mathcal{S}_2^0(\mathcal{M})$) on triangular meshes $\mathcal{M}$, obtained by regular refinement
- "Exact" evaluation of bilinear form by very high order quadrature : $a_h = a$
- ℓ_h from one point quadrature rule (2.7.5.36) of order 2 → midpoint QR



← midpoint QR

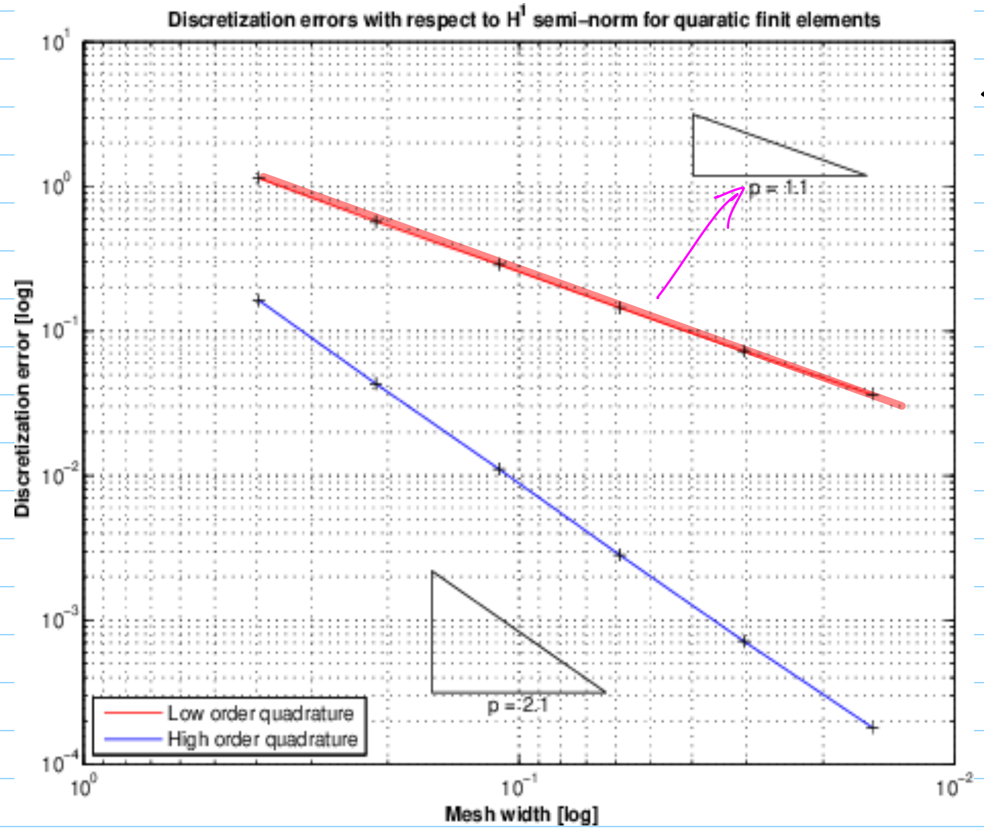
How to show this ?

MP-QR is exact for integrands $\in \mathcal{P}_1(\mathbb{R}^2)$:

[$\Leftrightarrow$ integrates $\lambda_1, \lambda_2, \lambda_3$ exactly]

Which cvg. in terms of $h_\mathcal{M} \to 0$ will hold without perturbations. [in energy norm = H^1-seminorm]

Sect 5.3 : $|u - u_h|_{H^1(\Omega)} = O(h_\mathcal{M}^2)$ as $h_\mathcal{M} \to 0$



$\triangle$ Alg. cvg. in $h_\mathcal{M}$, but with rate 1

$\Downarrow$

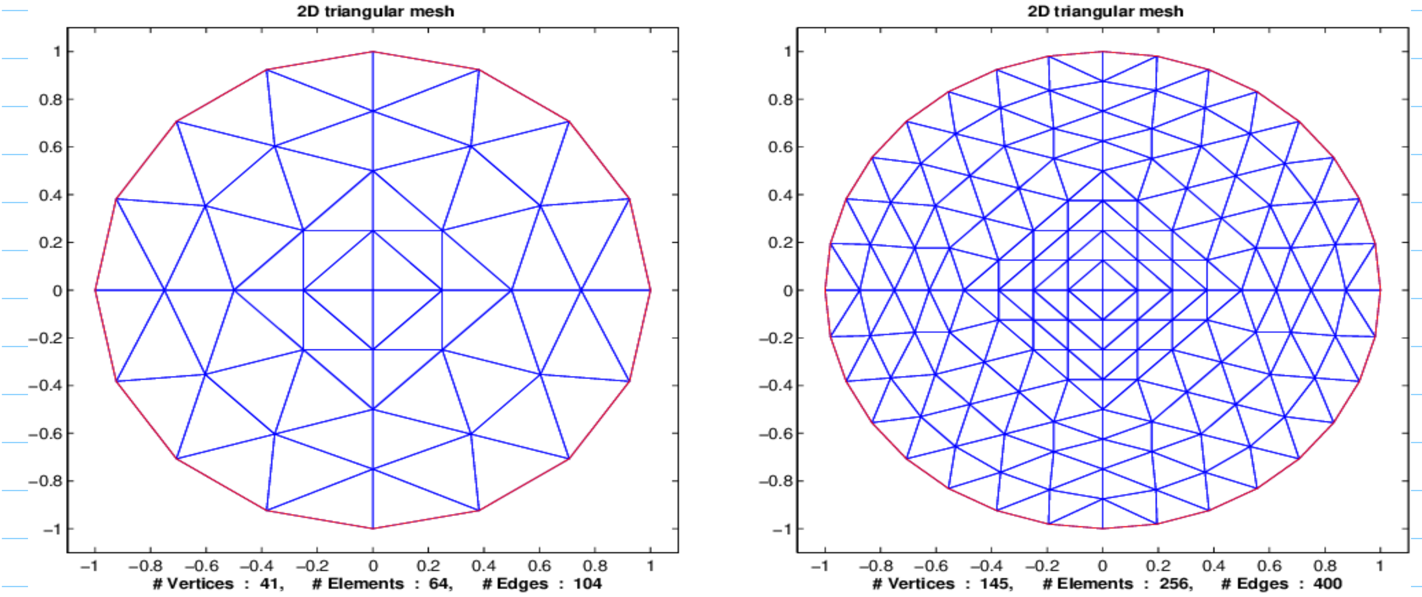
No admissible var. crime !

3

Finite element theory tells us that the above guideline can be met, if the local numerical quadrature rule has sufficiently high order. The quantitative results can be condensed into the following rules of thumb:

$\|u-u_h\|_1 = O(h^p_{\mathcal{M}})$ at best ▶	Quadrature rule of order $2p-1$ sufficient for right hand side functional ℓ_h.
$\|u-u_h\|_1 = O(h^p_{\mathcal{M}})$ at best ▶	Quadrature rule of order $2p-1$ sufficient for bilinear form a_h.

5.5.2. Boundary Approximation



Linearly boundary fitted unstructured triangular meshes of $\Omega = B_1(0)$.

[Meshes created by regular ref. w/ Gmsh]

Experiment 3.5.2.1:

$\Delta u = f$ in Ω

$u = 0$ on $\partial\Omega$

↳ smooth solution

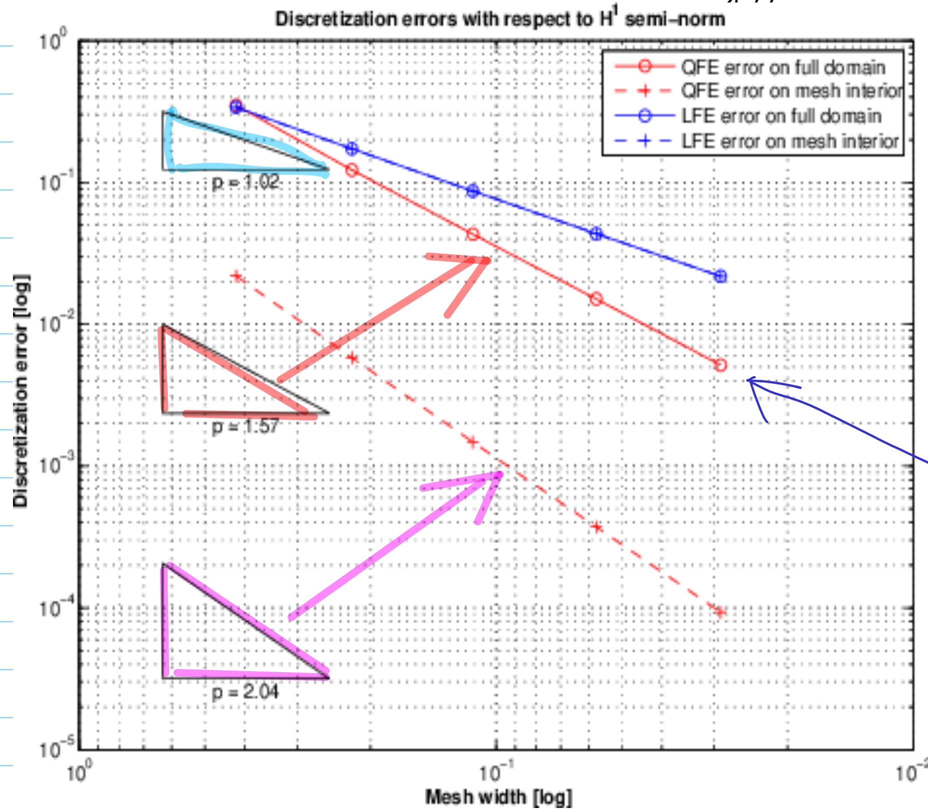
Setting: $\Omega := B_1(0) := \{\mathbf{x} \in \mathbb{R}^2 \colon |\mathbf{x}| < 1\}$, $u(r,\varphi) = \cos(r\pi/2)$ (polar coordinates)

➢ $f = \frac{\pi}{2r}\sin(r\pi/2) + \frac{\pi}{2}\cos(r\pi/2)$

- Sequences of unstructured triangular meshes $\mathcal{M}$ obtained by regular refinement (of coarse mesh with 4 triangles) + linear boundary fitting.
- Galerkin FE discretization based on $V_h := \mathcal{S}^0_{1,0}(\mathcal{M})$ or $V_h := \mathcal{S}^0_{2,0}(\mathcal{M})$.
- Recorded: approximate norm $|u-u_h|_{1,\Omega_h}$, evaluated using numerical quadrature rule (2.7.5.37).

(FE solution extended beyond the domain covered by $\mathcal{M}$ ("mesh interior") to Ω ("full domain") by means of polynomial extrapolation.)

↳ energy norm



LFE: Alg. cvg. with rate 1

▷ polygonal boundary approximation acceptable

$|u - \tilde{u}_h|_{H^1(\Omega)}$

↑

error computed for extrapolated FE solution on disk

Rule of thumb deduced from sophisticated finite element theory:

If $V_{0,h} = \mathcal{S}_p^0(\mathcal{M})$, use boundary fitting with polynomials of degree p.

See Ch. 2 for details

⑤ # Review questions 3.5.2.2 :

A :

The local 2D trapezoidal rule on a triangular mesh $\mathcal{M}$ of a domain $\Omega \subset \mathbb{R}^2$

$$\int_\Omega \varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \sum_{K\in\mathcal{M}} \tfrac{1}{3}|K| \sum_{\ell=1}^{3} \varphi(\boldsymbol{a}_K^\ell)\,,$$

is used to approximate the right-hand-side functional $\ell(v) := \int_\Omega f(\boldsymbol{x})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, $f \in C^0(\overline{\Omega})$, for a second-order elliptic boundary value problem, that is we use the perturbed functional

$$\ell_h(v) = \sum_{\boldsymbol{p}\in\mathcal{N}(\mathcal{M})} \omega_{\boldsymbol{p}} f(\boldsymbol{p}) v(\boldsymbol{p})\,,\quad \omega_{\boldsymbol{p}} \in \mathbb{R}\,, \tag{3.5.2.4}$$

where $\mathcal{N}(\mathcal{M})$ is the set of nodes of $\mathcal{M}$.

- What are the weights $\omega_{\boldsymbol{p}}$?
- Is the functional ℓ_h from (3.5.2.4) bounded on $H^1(\Omega)$?
- Is the functional ℓ_h from (3.5.2.4) continuous on $\mathcal{S}_1^0(\mathcal{M})$?

B :

We approximate the right-hand-side functional $\ell(v) := \int_\Omega f(\boldsymbol{x})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, $f \in C^\infty(\overline{\Omega})$, for a linear variational problem on $H_0^1(\Omega)$ by

$$\ell_h(v) := \int_\Omega (\mathsf{I}_p f)(\boldsymbol{x})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,,\quad v \in H_0^1(\Omega)\,, \tag{3.5.2.4}$$

where $\mathsf{I}_p : C^0(\overline{\Omega}) \to \mathcal{S}_p^0(\mathcal{M})$, $p \in \mathbb{N}$, is the standard nodal interpolation operator, $\mathcal{M}$ a triangular mesh.

- Is ℓ_h from (3.5.2.4) bounded on $H_0^1(\Omega)$?
- Predict the asymptotic dependence of the quantities

$$\delta(\mathcal{M}) := \sup_{v\in H_0^1(\Omega)} \frac{|(\ell-\ell_h)(v)|}{\|v\|_{H^1(\Omega)}}$$

on the meshwidth $h_\mathcal{M}$ on sequences of meshes obtained by uniform regular refinement.

One of the following results can help you answer the second question:

Theorem 3.3.5.6. Best approximation error estimates for Lagrangian finite elements

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded polygonal/polyhedral domain equipped with a mesh $\mathcal{M}$ consisting of simplices or parallelepipeds. Then, for each $k \in \mathbb{N}$, there is a constant $C > 0$ depending only on k and the shape regularity measure $\rho_\mathcal{M}$ such that

$$\inf_{v_h\in\mathcal{S}_p^0(\mathcal{M})} \|u - v_h\|_{H^1(\Omega)} \le C\left(\frac{h_\mathcal{M}}{p}\right)^{\min\{p+1,k\}-1} \|u\|_{H^k(\Omega)} \quad \forall u \in H^k(\Omega)\,. \tag{3.3.5.7}$$

Theorem 3.5.2.5. $H^1(\Omega)$-Norm interpolation error estimates for Lagrangian finite elements

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded polygonal/polyhedral domain equipped with a mesh $\mathcal{M}$ consisting of simplices or parallelepipeds. Then, for each $k \in \mathbb{N}$, $k \ge 2$, $p \in \mathbb{N}$ there is a constant $C > 0$ depending only on k, the polynomial degree *p, and the shape regularity measure $\rho_\mathcal{M}$ such that*

$$\|u - \mathsf{I}_p u\|_{H^1(\Omega)} \le C\,h_\mathcal{M}^{\min\{p+1,k\}-1} \|u\|_{H^k(\Omega)} \quad \forall u \in H^k(\Omega)\,, \tag{3.3.5.7}$$

where $\mathsf{I}_p : C^0(\overline{\Omega}) \to \mathcal{S}_p^0(\mathcal{M})$ is a nodal interpolation operator.

Theorem 3.5.2.6. $L^2(\Omega)$-Norm interpolation error estimates for Lagrangian finite elements

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded polygonal/polyhedral domain equipped with a mesh $\mathcal{M}$ consisting of simplices or parallelepipeds. Then, for each $k \in \mathbb{N}$, $k \ge 2$, $p \in \mathbb{N}$ there is a constant $C > 0$ depending only on k, the polynomial degree *p, and the shape regularity measure $\rho_\mathcal{M}$ such that*

$$\|u - \mathsf{I}_p u\|_{L^2(\Omega)} \le C\,h_\mathcal{M}^{\min\{p+1,k\}} \|u\|_{H^k(\Omega)} \quad \forall u \in H^k(\Omega)\,, \tag{3.3.5.7}$$

where $\mathsf{I}_p : C^0(\overline{\Omega}) \to \mathcal{S}_p^0(\mathcal{M})$ is a nodal interpolation operator.

C :

Which of the following practices is a **variational crime** (V.C.)?

1. The use of a Lagrangian finite elements of insufficient polynomial degree for a second-order elliptic boundary value problem

   ○ Definitely a V.C. ○ Not necessarily a V.C.

2. Computing the finite element solution of a BVP posed on $\Omega \subset \mathbb{R}^2$ on a mesh $\mathcal{M}$ for which

$$\bigcup\{\overline{K} : K \in \mathcal{M}\} \neq \overline{\Omega}\ ?$$

   ○ Definitely a V.C. ○ Not necessarily a V.C.

3. The use of the 2D trapezoidal rule for computing the element vectors for the right-hand-side functional $v \mapsto \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, $f \in C^0(\overline{\Omega})$.

   ○ Definitely a V.C. ○ Not necessarily a V.C.

4. The use of global shape functions that fail to satisfy the cardinal basis property with respect to a set of interpolation nodes

   ○ Definitely a V.C. ○ Not necessarily a V.C.

⚠ This list of review questions may not be complete. Additional review questions may be provided in the lecture document.