ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 3.6.1: Linear Output Functionals

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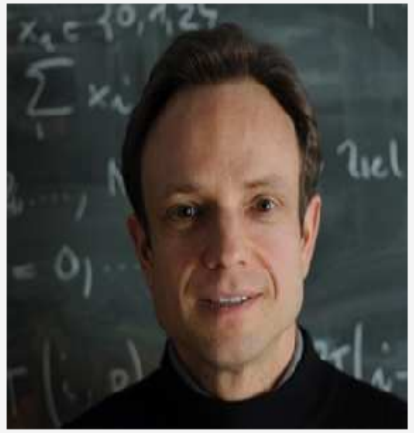

Dependency. [Lecture → ??], [Lecture → Section 3.1], [Lecture → Section 3.3.5], and [Lecture → Section 3.4]

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

III FEM: Convergence & Accuracy

3.6. FEM: Duality Techniques for Error Estimation

→ beyond energy norms

§3.6.1.1: Setting:

"Nice" LVP:

$$u \in V_0: \quad a(u,v) = \ell(v) \quad \forall v \in V_0$$

↳ s.p.d. ⇒ $\|\cdot\|_a$

Galerkin discr. with $V_{0,h} \subset V_0$

▷ DVP:

$$u_h \in V_{0,h}: \quad a(u_h,v_h) = \ell(v_h) \quad \forall v_h \in V_{0,h}$$

(2)

3.6.1 Linear Output Functionals

New twist: Now we are interested mainly/only in the *number* $F(u)$, where

$$\boxed{F : V_0 \mapsto \mathbb{R}}$$ is a given output functional.

Mathematical terminology: functional $\hat{=}$ mapping from a function space into $\mathbb{R}$

We compute an approximation $F(u_h) \approx F(u)$

Focus on a special class of OF:

Assumption 3.6.1.3. Linearity of output functional

The output functional F is a linear form (→ Def. 0.2.1.7) on V_0

Assumption 3.6.1.4. Continuity of output functional → Def. 1.2.3.45

The output functional is continuous w.r.t. the energy norm in the sense that

$$\exists C_f > 0: \quad |F(v)| \leq C_f \|v\|_a \quad \forall v \in V_0 .$$

Simple output error estimate

$$|F(u) - F(u_h)| = |F(u - u_h)| \leq C_F \|u - u_h\|_a$$

Exp 3.6.1.6 :

We consider a simple non-dimensional heat conduction model (→ Section 1.6), scaled heat conductivity $\kappa \equiv 1$, on the domain $\Omega =]0,1[^2$, with fixed temperature $u = 0$ on $\partial\Omega$:

$$-\Delta u = f \quad \text{in } \Omega \quad , \quad u = 0 \quad \text{on } \partial\Omega .$$

We choose the heat source functio $f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$, $\begin{bmatrix} x \\ y \end{bmatrix} \in \Omega$, such that we obtain the exact olution $u(x,y) = \sin(\pi x)\sin(\pi y)$. We are interested in the

$$\text{mean temperature:} \quad F(u) := \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x .$$

↳ linear & continuous

Details of finite element Galerkin discretization:

- Sequence of triangular meshes $\mathcal{M}$ created by regular refinement.
- Galerkin discretization: (I) $V_{0,h} := \mathcal{S}^0_{1,0}(\mathcal{M})$ (linear Lagrangian finite elements → Section 2.4), $p=1$
  (II) $V_{0,h} := \mathcal{S}^0_{2,0}(\mathcal{M})$ (quadratic Lagrangian finite elements → Ex. 2.6.1.2). $p=2$
- Quadrature rule (2.7.5.37) of order 6 for assembly of right hand side vector (more than sufficiently accurate according to the guidelines of Section 3.5.1)

Expected:

$$|F(u) - F(u_h)| \leq C\,|u - u_h|_{H^1(\Omega)} = O(h_{\mathcal{M}}^p)$$

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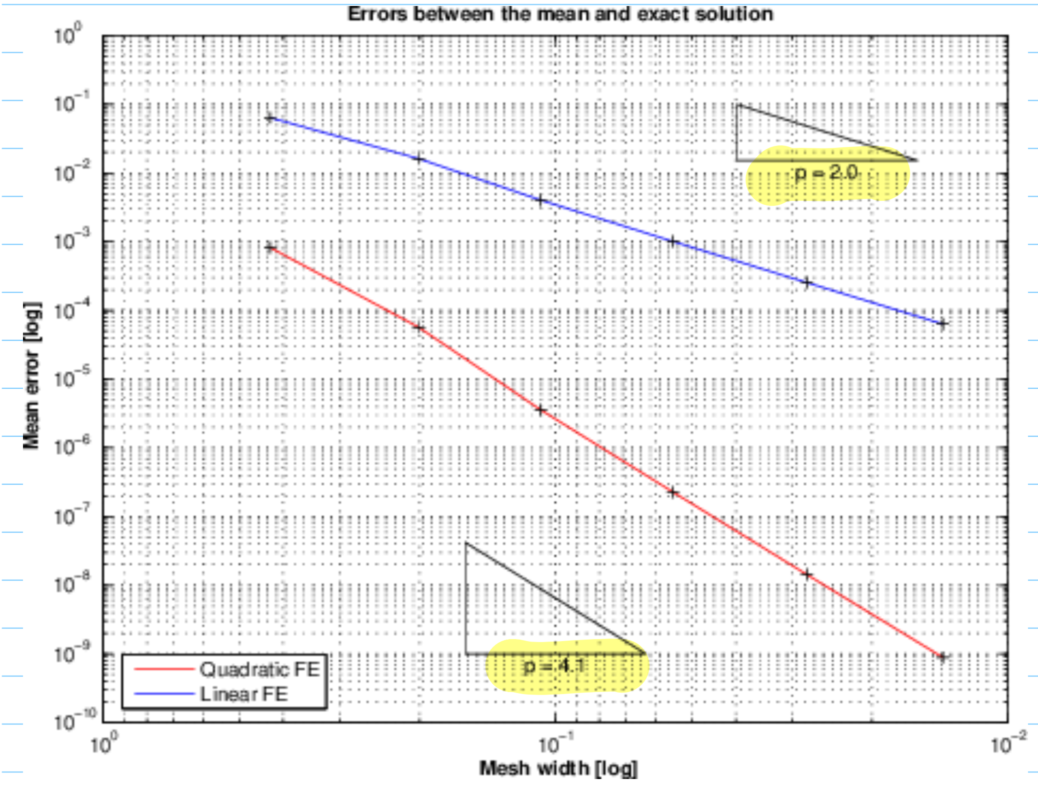

▷ alg. cvg. of output error

rate $2p$!

Explanation by duality estimate

Trick: Consider dual variational problem

$$\forall g_F \in V_0 : \quad a(v, g_F) = F(v) \quad \forall v \in V_0 \quad (*)$$

↳ dual solution

$$F(u) - F(u_h) = F(u-u_h) \overset{(*)}{=} a(u-u_h, g_F - w_h)$$

$v \to u - u_h \in V_0$ $\qquad \forall w_h \in V_{0,h}$

by Galerkin orthogonality $a(u-u_h, w_h) = 0 \ \forall w_h \in V_{0,h}$

▷ By Cauchy-Schwarz for $a(\cdot,\cdot)$

$$|F(u) - F(u_h)| \leq \|u - u_h\|_a \cdot \inf_{w_h \in V_{0,h}} \|g_F - w_h\|_a$$

Theorem 3.6.1.7. Duality estimate for linear functional output

Define the dual solution $g_F \in V_0$ to F as solution of the dual variational problem

$$g_F \in V_0: \quad \mathsf{a}(v, g_F) = F(v) \quad \forall v \in V_0 .$$

Then

$$|F(u) - F(u_h)| \leq \|u - u_h\|_{\mathsf{a}} \cdot \inf_{v_h \in V_{0,h}} \|g_F - v_h\|_{\mathsf{a}} . \qquad (3.6.1.8)$$

energy of disc. error $\qquad$ best approximation error for dual solution

Back to mean temperature: $F(v) = \frac{1}{|\Omega|}\int_\Omega 1 \cdot v \, dx$

⇓

[BVP in strong form]: $\quad -\Delta g_F = \frac{1}{|\Omega|}$ in Ω, $\quad g_F = 0$ on $\partial\Omega$

Crucial: Smoothness of g_F

Theorem 3.4.0.10. Elliptic lifting theorem on convex domains

If $\Omega \subset \mathbb{R}^d$ *convex*, $u \in H_0^1(\Omega)$, $\Delta u \in L^2(\Omega)$ $\Rightarrow$ $u \in H^2(\Omega)$.

& $\|u\|_{H^2(\Omega)} \leq C(\Omega) \|\Delta u\|_{L^2(\Omega)}$

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In Exp.: $g_F \in H^2$

$$\inf_{w_h \in S^0_{1,0}(\mathcal{M})} |g_F - w_h|_{H^1(\Omega)} = O(h_{\mathcal{M}}^1)$$

$$F(u) - F(u_h) \leq C\, h_{\mathcal{M}} \cdot h_{\mathcal{M}} = O(h_{\mathcal{M}}^2)$$

from $\|u - u_h\|_a$

$p = 2$: In our setting $g_F \in H^3(\Omega)$

Then apply $\triangleright$ $O(h_{\mathcal{M}}^2)$-error for g_F

Theorem 3.3.5.6. Best approximation error estimates for Lagrangian finite elements

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded polygonal/polyhedral domain equipped with a mesh $\mathcal{M}$ consisting of simplices or parallelepipeds. Then, for each $k \in \mathbb{N}$, there is a constant $C > 0$ depending only on k and the shape regularity measure $\rho_{\mathcal{M}}$ such that

$$\inf_{v_h \in \mathcal{S}^0_p(\mathcal{M})} \|u - v_h\|_{H^1(\Omega)} \leq C \left(\frac{h_{\mathcal{M}}}{p}\right)^{\min\{p+1,k\}-1} \|u\|_{H^k(\Omega)} \quad \forall u \in H^k(\Omega)\,. \qquad (3.3.5.7)$$

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Review questions 3.6.1.12 :

[Please try to solve without aids]

A :

Prove that the "mean temperature functional"

$$F : H^1(\Omega) \to \mathbb{R} \quad , \quad v \mapsto F(v) := \frac{1}{|\Omega|} \int_\Omega v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

is continuous/bounded.

Definition 1.2.3.42. Continuity of a linear form and bilinear form

Consider a normed vector space V_0 with norm $\|\cdot\|$. A linear form $\ell : V_0 \to \mathbb{R}$ (→ Def. 0.3.1.4) is continuous or bounded on V_0, if

$$\exists C > 0: \quad |\ell(v)| \leq C\|v\| \quad \forall v \in V_0 \, .$$

A bilinear form $\mathrm{a} : V_0 \times V_0 \to \mathbb{R}$ (→ Def. 0.3.1.4) on V_0 is continuous, if

$$\exists K > 0: \quad |\mathrm{a}(u, v)| \leq K\|u\|\|v\| \quad \forall u, v \in V_0 \, .$$

B :

Let $u \in V_0$ denote that solution of linear variational problem

$$u \in V_0: \quad \mathrm{a}(u, v) = \ell(v) \quad \forall v \in V_0 \, , \tag{2.2.0.2}$$

posed on a vector space V_0, and with an s.p.d. bilinear form $\mathrm{a} : V_0 \times V_0 \to \mathbb{R}$ inducing the energy norm $\|\cdot\|_{\mathrm{a}}$, $\ell : V_0 \to \mathbb{R}$ a linear functional, continuous/bounded with respect to $\|\cdot\|_{\mathrm{a}}$.
Write $F : V_0 \to \mathbb{R}$ for a linear functional that is continuous/bounded with respect to $\|\cdot\|_{\mathrm{a}}$, and $u_h \in V_{0,h}$ for the Galerkin solution of (2.2.0.2), $V_{0,h} \subset V_0$ some subspace
Give a three-line proof of the following **duality estimate**.

Theorem 3.6.1.7. Duality estimate for linear functional output

Define the dual solution $g_F \in V_0$ to F as solution of the dual variational problem

$$g_F \in V_0: \quad \mathrm{a}(v, g_F) = F(v) \quad \forall v \in V_0 \, .$$

Then

$$|F(u) - F(u_h)| \leq \|u - u_h\|_{\mathrm{a}} \inf_{v_h \in V_{0,h}} \|g_F - v_h\|_{\mathrm{a}} \, . \tag{3.6.3.14}$$