ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 3.6.2: Case Study: Computation of Boundary Fluxes with FEM

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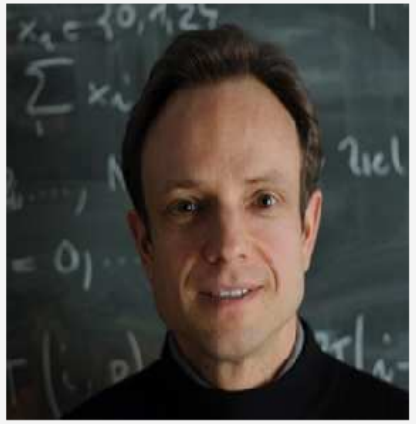


Dependency. [Lecture → Section 3.6.1]

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

III FEM: Convergence & Accuracy

3.6. FEM: Duality Techniques for Error Estimation

3.6.2. Case Study: Computation of Boundary Fluxes with FEM

Model problem (process engineering):

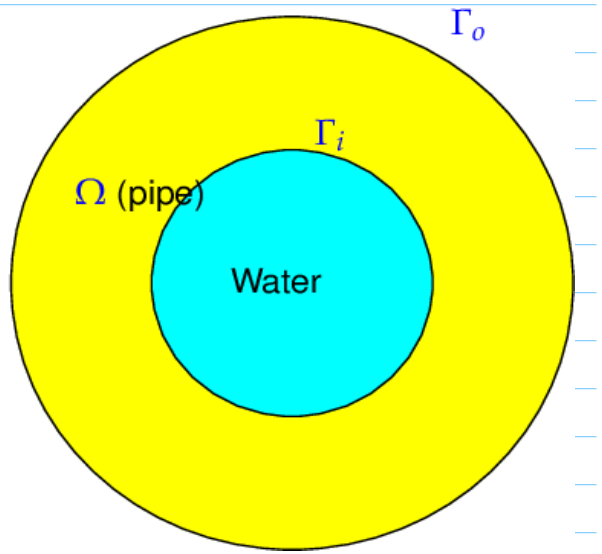
Long pipe carrying turbulent flow of coolant (water)

$\Omega \subset \mathbb{R}^2$: cross-section of pipe

$\kappa = 1$: (scaled) heat conductivity of pipe material (assumed homogeneous, $\kappa = \mathrm{const}$)

Assumption: Constant temperatures u_o, , u_i at outer/inner wall Γ_o, Γ_i of pipe

Task: Compute heat flow pipe → water



Mathematical model: elliptic boundary value for stationary heat conduction (→ Section 1.6)

$$-\operatorname{div}(\kappa\,\mathbf{grad}\,u) = 0 \quad \text{in } \Omega \quad , \quad u = u_x \quad \text{on } \Gamma_x,\ x \in \{i, o\} \,. \tag{3.6.2.1}$$

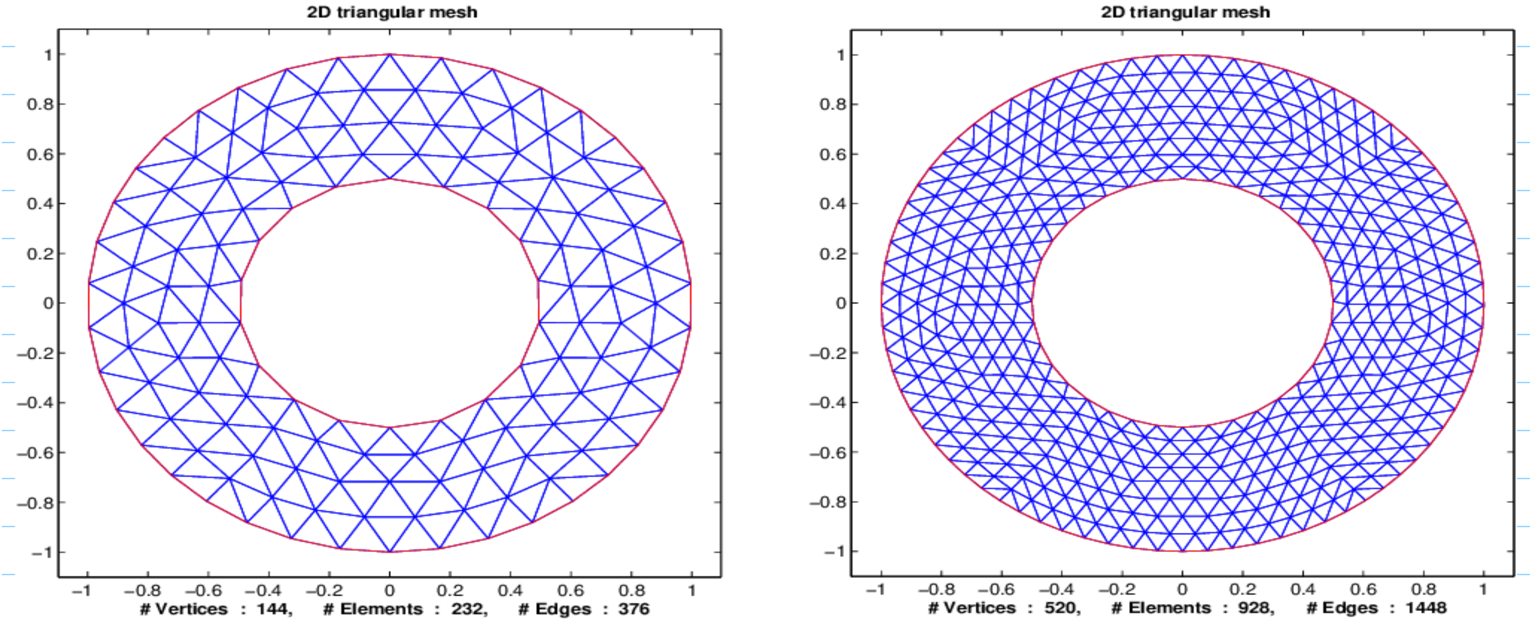
Heat flux through Γ_i:
$$J(u) := \int_{\Gamma_i} \kappa\,\mathbf{grad}\,u \cdot \boldsymbol{n}\,\mathrm{d}S \,. \tag{3.6.2.2}$$

(2) Exp. 3.6.2.3:

- Setting: model problem "heat flux pipe → water", see (3.6.2.1) and Fig. 213.
- Linear output functional from (3.6.2.2)
  - Domain $\Omega = B_{R_o}(0) \setminus B_{R_i}(0) := \{\mathbf{x} \in \mathbb{R}^2 \colon R_i < |\mathbf{x}| < R_o\}$ with $R_o = 1$ and $R_i = 1/2$
  - Dirichlet boundary data $u_i = 60°\mathrm{C}$ on Γ_i, $u_o = 10°\mathrm{C}$ on Γ_o, heat source $f \equiv 0$, heat conductivity $\kappa \equiv 1$.

➢ Exact solution: $u(r,\varphi) = C_1 \ln(r) + C_2$, with $C_1 := (u_o - u_i)/(\ln R_i - \ln R_o)$,

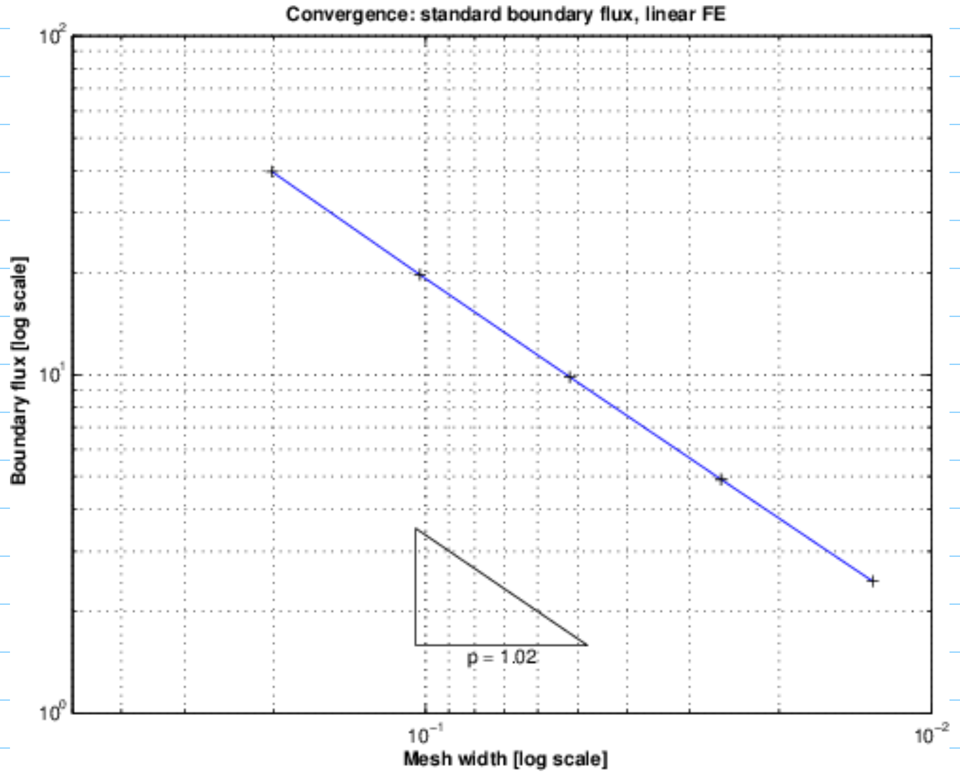
➢ Exact heat flux: $J = 2\pi\kappa C_1$, $C_2 := (\ln R_o u_i - \ln R_i u_o)/(\ln R_i - \ln R_o)$.



Unstructured triangular meshes for $\Omega = B_1(0) \setminus B_{1/2}(0)$ (two coarsest specimens).

- Sequences of unstructured triangular meshes $\mathcal{M}$ obtained by regular refinement of coarse mesh (from grid generator).
- Galerkin FE discretization based on $V_{0,h} := \mathcal{S}^0_{1,0}(\mathcal{M})$.
- Approximate evaluation of $\mathsf{a}(u_N, v_N)$, $f(v_N)$ by six point quadrature rule (2.7.5.37) ("overkill quadrature", see Section 3.5.1)
- Approximate evaluation of $J(u_N)$ by 4 point Gauss-Legendre quadrature rule on boundary edges of $\mathcal{M}$.
- Linear boundary approximation (circle replaced by polygon).
- Recorded: errors $|J(u) - J(u_h)|$ on sequence of meshes.

Expect (by "duality estimates")
↳ $|J(u) - J(u_h)| = O(h_{\mathcal{M}}^2)$



◁ Output error, alg. cvg. Rate only 1

§3.6.2.4:

Note: $J(v) = \int_{\Gamma_i} \mathbf{grad}\, v \cdot \mathbf{n}\, dS$ is not continuous wrt. to $\|\cdot\|_a$:

We can find $v \in H^1(\Omega)$: "$J(v) = \infty$"

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Trick: Replace J with continuous $J^*: V_0 \to \mathbb{R}$ satisfying $J(u) = J^*(u)$

↑ exact solution of BVP

Intuition: J^* via cut-off function:

$$\psi \in H^1(\Omega)\,,\quad \psi_{|\Gamma_i} \equiv 1\,,\quad \psi_{|\Gamma_0} \equiv 0$$

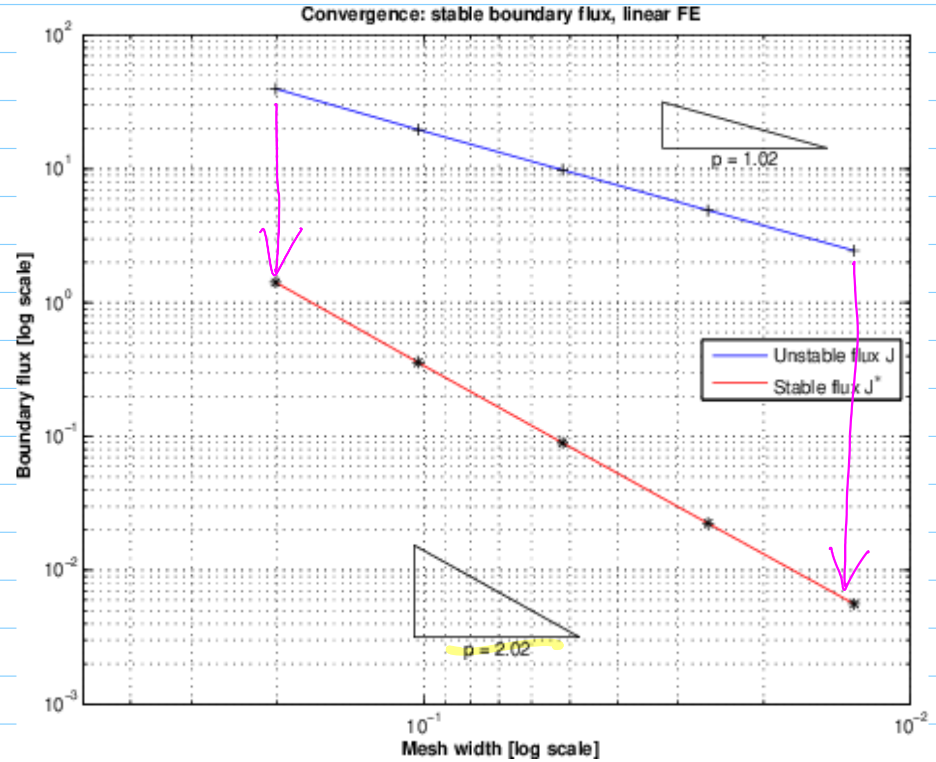
$$J(u) = \int_{\Gamma_i} \psi \cdot \kappa\, \mathbf{grad}\, u \cdot n \, dS \overset{\text{i.b.p.}}{=} \int_\Omega \mathbf{grad}\,\psi \cdot \kappa\, \mathbf{grad}\, u + \psi\, \underbrace{\operatorname{div}(\kappa\, \mathbf{grad}\, u)}_{=0} \, dx$$

$$\triangleright \quad J^*(v) = \int_\Omega \kappa\, \mathbf{grad}\,\psi \cdot \mathbf{grad}\, u \, dx$$

↳ continuous on $H^1(\Omega)$ (by Cauchy-Schwarz)

Exp cnt'd:

- Galerkin FE discretization based on $V_{0,h} := \mathcal{S}^0_{1,0}(\mathcal{M})$ or $V_{0,h} := \mathcal{S}^0_{2,0}(\mathcal{M})$.
- Approximate evaluation of $J^*(u_N)$ by six point quadrature rule (2.7.5.37) ("overkill quadrature", see Section 3.5.1)
- Cut-off function with linear decay in radial direction: $\psi(\mathbf{x}) = 2\|\mathbf{x}\| - 1$, $\psi \in C^\infty(\overline{\Omega})$.
- Recorded: output errors $|J(u) - J(u_h)|$ and $|J(u) - J^*(u_N)|$.



◁ Output error for J^*, alg. cvg.
Rate 2
+ Substantially better error constants

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Review questions 3.6.2.13 :

[Unless told otherwise, you should answer them without resorting to lecture documents]

A :

We consider the Galerkin discretization of the Dirichlet BVP

$$-\Delta u = f \quad \text{in}\ \Omega \quad , \quad u = 0 \quad \text{on}\ \partial\Omega \,,$$

$\Omega =]0,1[^2$, by means of degree-2 Lagrangian finite elements, $V_{0,h} = \mathcal{S}_2^0(\mathcal{M})$, on sequences of triangular meshes created by uniform regular refinement.
For the "mean temperature functional"

$$F : H^1(\Omega) \to \mathbb{R} \quad , \quad v \mapsto \frac{1}{|\Omega|}\int_\Omega v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

determine the rate of algebraic convergence $F(u_h) \to F(u)$ in terms of meshwidth $h_{\mathcal{M}} \to 0$, when $u_h \in \mathcal{S}_2^0(\mathcal{M})$ is the finite-element solution.

Hint (→ Rem. 3.6.1.11): If $\Omega =]0,1[^2$, $u \in H_0^1(\Omega)$, $-\Delta u = f$, $f \in H^1(\Omega)$, then $u \in H^3(\Omega)$ and there is a constant $C > 0$ such that $\|u\|_{H^3(\Omega)} \leq C\|f\|_{H^1(\Omega)}$.

B :

We consider the Dirichlet boundary value problem

$$-\Delta u = 0 \quad \text{in}\ \Omega \quad , \quad u = g \quad \text{on}\ \partial\Omega \,,$$

for given bundary data $g \in C^0(\partial\Omega)$ and on a domain $\Omega \subset \mathbb{R}^2$. In Exp. 3.6.2.7 we studied the regularized boundary flux functional

$$J^*(u) := \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, \psi \,\mathrm{d}\boldsymbol{x} \qquad \psi \in H^1(\Omega) \,.$$

State the boundary value problem that is solved by the dual solution g_F induced by this regularized output functional J^* in strong (PDE) form.

C : → Use lecture notes

If in **Exp. 3.6.2.7** we change the computational domain from an annulus to $\Omega :=]-1,1[^2 \setminus [-\frac{1}{2},\frac{1}{2}]^2$. Speculate how this would affect the convergence rates observed in Fig. 235.

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D:

The components of the linear variational problem

$$u \in V_0: \quad \mathsf{a}(u,v) = \ell(v) \quad \forall v \in V_0 \,, \tag{2.2.0.2}$$

on the vector space V_0, are supposed to satisfies the "usual assumptions". Thus, it possesses a unique solution $u \in V_0$ for every ($\|\cdot\|_{\mathsf{a}}$-continuous) right-hand-side linear functional ℓ. We perform its Galerkin discretization based on a subspace $V_{0,h} \subset V_0$.
For a continuous output functional $F : V_0 \to \mathbb{R}$ let $g_{F,h}$ be the Galerkin solution of (2.2.0.2) with ℓ replaced with F. What does $\ell(g_{F,h})$ give you?

E:

Let

$$u \in H^1(\Omega): \quad \mathsf{a}(u,v) = \int_\Omega f_k(\boldsymbol{x})\, v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \forall v \in H^1(\Omega) \,, \tag{3.6.3.15}$$

be the variational formulation of a second-order elliptic boundary value problem, $\mathsf{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ an symmetric positive definite bilinear form, $f_k \in L^2(\Omega)$.
We are interested in approximately evaluating $F(u_k)$, u_k the solution of (3.6.3.15), for a bounded *linear functional* $F : H^1(\Omega) \to \mathbb{R}$ and for *many* different and unrelated source functions $f_k \in L^2(\Omega)$, $k = 1, \dots, m$, $m \in \mathbb{N}$. We rely on finite element Galerkin approximation based on piecewise linear Lagrangian finite elements, $V_{0,h} = \mathcal{S}_1^0(\mathcal{M})$, producing the Galerkin solutions $u_{k,h} \in \mathcal{S}_1^0(\mathcal{M})$.
Outline an algorithm for computing $F(u_{k,h})$, $k = 1, \dots, m$, which involves a minimal number of solves of sparse linear systems of equations.

⚠ This list of review questions may not be complete. Additional review questions may be provided in the lecture document.