ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 3.6.3: Lagrangian FEM: L^2-Estimates

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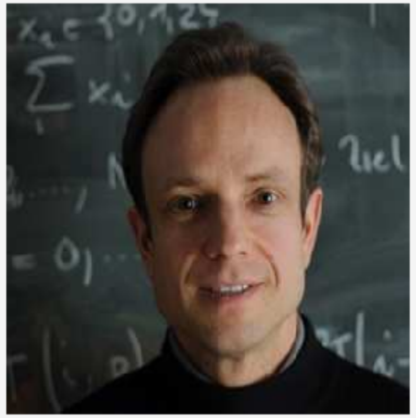
Date: March 21, 2019



Prerequisites.

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Dependency. [Lecture → Section 3.6.1]

⚠ Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

III FEM: Convergence & Accuracy

3.6. FEM: Duality Techniques for Error Estimation

3.6.3. Lagrangian FEM: L^2-Estimates

Goal: Study cvg. $\|u-u_h\|_{L^2(\Omega)}$ as $h_{\mathcal{M}} \to 0$

▷ Remember:

Theorem 3.3.2.21. Error estimate for piecewise linear interpolation

For any $u \in C^2(\bar{\Omega})$ and 2D piecewise linear interpolation $\mathsf{I}_1 : C^0(\overline{\Omega}) \to \mathcal{S}_1^0(\mathcal{M})$, $\mathcal{M}$ a triangular mesh, holds

$$\|u - \mathsf{I}_1 u\|_{L^2(\Omega)} \leq \sqrt{\tfrac{3}{8}}\, h_{\mathcal{M}}^2 \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)}, = O(h_{\mathcal{M}}^2)$$

$$\|\mathbf{grad}(u - \mathsf{I}_1 u)\|_{L^2(\Omega)} \leq \sqrt{\tfrac{3}{24}}\, \rho_{\mathcal{M}} h_{\mathcal{M}} \left\| \left\| D^2 u \right\|_F \right\|_{L^2(\Omega)}. = O(h_{\mathcal{M}})$$

where $h_{\mathcal{M}}$ denotes the mesh width (→ Def. 3.2.1.4) and $\rho_{\mathcal{M}}$ the shape regularity measure (→ Def. 3.3.2.20) of $\mathcal{M}$.

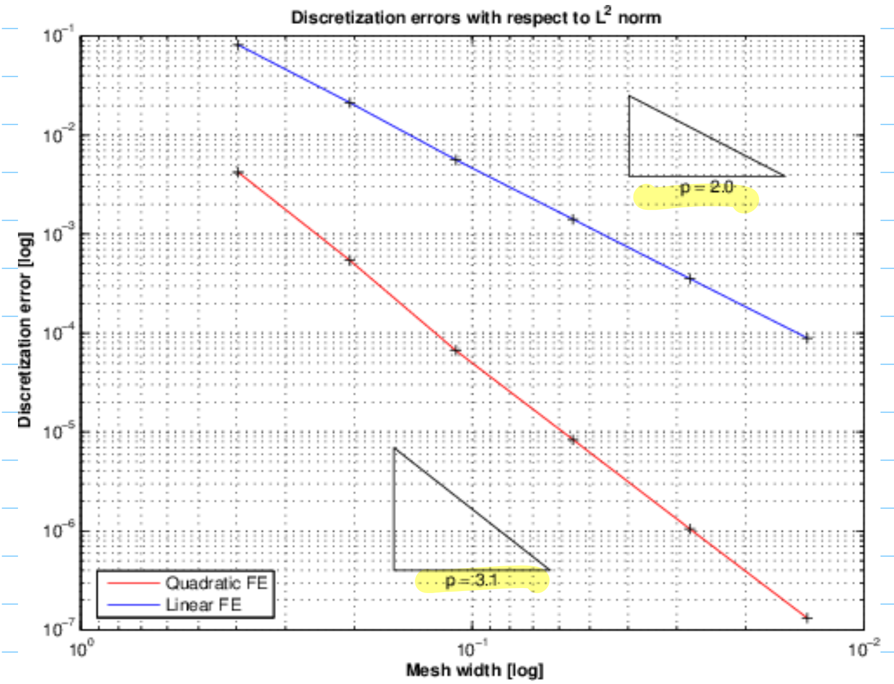
Exp 3.2.3.8 (Convergence of linear & quadratic Lagr. FE in L^2-norm)

$V_0 = H_0^1(\Omega)$, $a(u,v) = \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \, dx$

Setting: $\Omega =]0,1[^2$, ..., $f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$, $(x,y)^\top \in \Omega$

➢ $u(x,y) = \sin(\pi x)\sin(\pi y)$.

- Sequence of triangular meshes $\mathcal{M}$, created by regular refinement.
- FE Galerkin discretization based on $\mathcal{S}^0_{1,0}(\mathcal{M})$ or $\mathcal{S}^0_2(\mathcal{M})$.
- Quadrature rule (2.7.5.37) for assembly of local load vectors ($\rightarrow$ Section 2.7.5).
- Approximate $L^2(\Omega)$-norm by means of quadrature rule (2.7.5.37).



Δ alg. cvg. of L^2-error

rate $p+1$

for $\mathcal{S}^0_{p,0}(\mathcal{M})$

However: Cannot conclude this directly from best approximation estimates!

Tool: Duality technique

Idea: Consider special continuous linear "output functional"

linear & H^1-continuous $\longrightarrow$ $F(v) := \int_\Omega v \cdot (u - u_h)\,\mathrm{d}x$!

▷ $F(u) - F(u_h) = \|u - u_h\|^2_{L^2(\Omega)}$

Apply:

Theorem 3.6.1.7. Duality estimate for linear functional output

Define the dual solution $g_F \in V_0$ to F as solution of the dual variational problem

$$g_F \in V_0: \quad \mathrm{a}(v, g_F) = F(v) \quad \forall v \in V_0 \,.$$

Then

$$|F(u) - F(u_h)| \leq \|u - u_h\|_{\mathrm{a}} \inf_{v_h \in V_{0,h}} \|g_F - v_h\|_{\mathrm{a}} \,. \qquad (3.6.1.8)$$

Dual BVP (strong form)

$$-\Delta g_F = u - u_h \text{ in } \Omega$$
$$g_F = 0 \text{ on } \partial\Omega$$

Needed: Information on smoothness of g_F

Assumption 3.6.3.6. 2-regularity of homogeneous Dirichlet problem

We assume that the homogeneous Dirichlet problem with coefficient κ is 2-regular on Ω: There is $C > 0$, which depends on Ω only such that

$$g_F \in H^1_0(\Omega),\ \mathrm{div}(\cancel{\kappa(\mathbf{x})}\,\mathbf{grad}\, g_F) \in L^2(\Omega) \Rightarrow g_F \in H^2(\Omega) \text{ and } \|g_F\|_{H^2(\Omega)} \leq C\|\mathrm{div}(\cancel{\kappa(\mathbf{x})}\,\mathbf{grad}\, g_F)\|_{L^2(\Omega)} \,.$$

[Ell. reg. th: holds for convex Ω]

Theorem 3.4.0.10. Elliptic lifting theorem on convex domains

If $\Omega \subset \mathbb{R}^d$ *convex,* $u \in H^1_0(\Omega)$, $\Delta u \in L^2(\Omega)$ $\Rightarrow$ $u \in H^2(\Omega)$.

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$$|F(u)-F(u_h)| \leq |u-u_h|_{H^1} \cdot \inf_{w_h \in V_{0,h}} |u-w_h|_{H^1}$$

Now use for $k=2$

Theorem 3.3.5.6. Best approximation error estimates for Lagrangian finite elements

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded polygonal/polyhedral domain equipped with a mesh $\mathcal{M}$ consisting of simplices or parallelepipeds. Then, for each $k \in \mathbb{N}$, there is a constant $C > 0$ depending only on k and the shape regularity measure $\rho_{\mathcal{M}}$ such that

$$\inf_{v_h \in \mathcal{S}_p^0(\mathcal{M})} \|u - v_h\|_{H^1(\Omega)} \leq C\left(\frac{h_{\mathcal{M}}}{p}\right)^{\min\{p+1,k\}-1} \|u\|_{H^k(\Omega)} \quad \forall u \in H^k(\Omega)\,. \tag{3.3.5.7}$$

$$\inf_{w_h \in V_{0,h}} |u-w_h|_{H^1} \leq C\left(h_{\mathcal{M}}/p\right) |g_F|_{H^2(\Omega)}$$

$$\text{Ass. } 3.6.3.6 \Rightarrow |g_F|_{H^2} \leq C \|u-u_h\|_{L^2}$$

$$\triangleright \quad \|u-u_h\|_{L^2}^2 = |F(u)-F(u_h)| \leq$$

$$\leq \underbrace{|u-u_h|_{H^1}}_{O(h_{\mathcal{M}}) \text{ for } u \in H^2(\Omega)} \cdot C\left(h_{\mathcal{M}}/p\right) \cancel{\|u-u_h\|_{L^2}}$$

$$= O(h_{\mathcal{M}}^2)$$

With $C > 0$ depending only on Ω, κ, and the shape regularity measure $\rho_{\mathcal{M}}$ we conclude

$$\boxed{\text{Ass. } 3.6.3.6 \;\Rightarrow\; \|u-u_h\|_{L^2(\Omega)} \leq C\,\frac{h_{\mathcal{M}}}{p}\|u-u_h\|_{H^1(\Omega)}}\,. \tag{3.6.3.10}$$

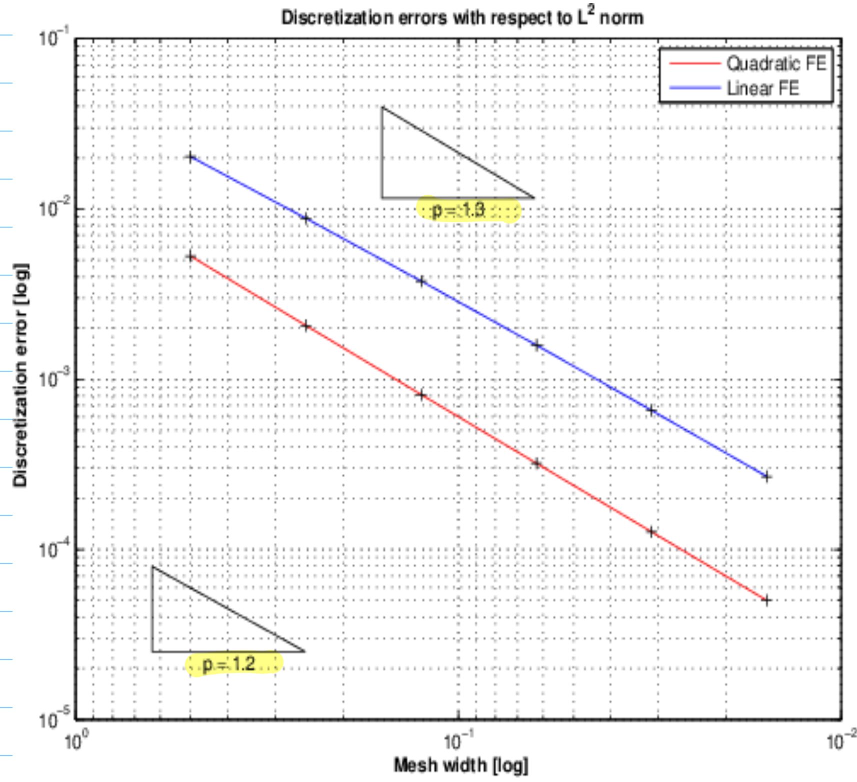
▶ for h-refinement: gain of one factor $O(h_{\mathcal{M}})$ (vs. $H^1(\Omega)$-estimates)

2-regularity necessary ?

corner singular

Setting: $\Omega =]-1,1[^2 \setminus ([0,1[\times]-1,0])$, $D \equiv 1$, $u(r,\varphi) = r^{2/3}\sin(2/3\varphi)$ (polar coordinates)
$f = 0$, Dirichlet data $g = u_{|\partial\Omega}$.
Finite element Galerkin discretization and evaluations as in Exp. 3.6.3.1.

→ L-shaped (non-convex) domain



alg. cvg. of L^2-error

rate ≈ 1.2

(for both $p=1,2$)

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Review questions 3.6.3.13 :

[Answer off the top of your head immediately after having studied the unit]

A :

In the setting of **Exp. 3.6.3.1** give an estimate (as sharp as possible) for

1. $\|u - u_h\|_{L^2(\Omega)}$
2. and $\|u - u_h\|_{L^2(\partial\Omega)}$,

where u_h stands for finite element solutions obtained with either piecewise linear or piecewise quadratic Lagrangian finite elements.

Theorem 1.9.0.10. Multiplicative trace inequality

$$\exists C = C(\Omega) > 0: \quad \|u\|^2_{L^2(\partial\Omega)} \leq C\|u\|_{L^2(\Omega)} \cdot \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\ .$$

B :

We consider the linear variational problem

$$u \in V_0: \quad \mathsf{a}(u, v) = \ell(v) \quad \forall v \in V_0\ ,$$

on a vector space V_0, and with an s.p.d. bilinear form $\mathsf{a} : V_0 \times V_0 \to \mathbb{R}$ inducing the energy norm $\|\cdot\|_{\mathsf{a}}$, $\ell : V_0 \to \mathbb{R}$ a linear functional, bounded with respect to $\|\cdot\|_{\mathsf{a}}$.
We are interested in a quadratic output functional ($\to$ Def. 1.2.3.2)

$$F(v) = \tfrac{1}{2}\mathsf{b}(v, v) - f(v)\ , \tag{3.6.3.16}$$

$\mathsf{b} : V_0 \times V_0 \to \mathbb{R}$ a symmetric bilinear form, $f : V_0 \to \mathbb{R}$ a linear form satisfying

$$\exists C > 0: \quad |\mathsf{b}(v, w)| \leq C\|v\|_{\mathsf{a}}\|w\|_{\mathsf{a}} \quad , \quad |f(v)| \leq C\|v\|_{\mathsf{a}} \quad \forall v, w \in V_0\ .$$

For the Galerkin solution $u_h \in V_{0,h}$, $V_{0,h} \subset V$ a subspace, establish the duality estimate

$$|F(u_h) - F(u)| \leq \|u - u_h\|_{\mathsf{a}} \inf_{v_h \in V_{0,h}} \|g - v_h\|_{\mathsf{a}}$$

and characterize $g \in V_0$.

This list of review questions may not be complete. Additional review questions may be provided in the lecture document.

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