ETH Lecture 401-0674-00L Numerical Methods for (Partial) Differential Equations

Course Video
Section 4.1: Finite-Difference Methods (FDMs)

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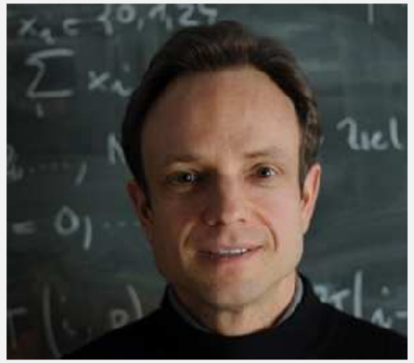


Prerequisites.

- From analysis: difference quotients

Dependencies. [Lecture → Section 1.5.1], [Lecture → Section 1.5.3], [Lecture → Section 2.3], [Lecture → Section 2.4]

Duration: 40 minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

IV. Beyond FEM
Alternative Discretizations

4.1. Finite-Difference Methods (FDM)

Construction of finite-difference methods (FDMs): Policy

The idea underlying finite-difference methods is to

(I) replace derivatives with **difference quotients**,

(II) which are anchored at the nodes/grid points of a (*structured*) mesh/grid,

(III) and access solution values only at other nodes/grid points.

FDM attack BVP in strong form

4.1.1. FDM for Two-Point BVPs

$$-\frac{d}{dx}\left(\sigma(x)\frac{du}{dx}(x)\right) = f(x) \quad \text{in} \quad]a,b[\quad , \quad u(a) = u_a \; , \quad u(b) = u_b \; . \qquad (1.5.1.16)$$

continuous

2 (i) Start with mesh/grid

$$\mathcal{V}(\mathcal{M}) := \{a = x_0 < x_1 < \cdots < x_{M-1} < x_M = b\} .$$

$h_j := x_j - x_{j-1}$

(ii) Derivatives $\longrightarrow$ difference quotients

$$\frac{du}{dx}(x_0) \approx \frac{u(x_0+h) - u(x_0-h)}{2h} , \quad \text{with span} \quad h > 0 . \tag{4.1.1.4}$$

$$-\frac{d}{dx}\left(\sigma(x)\frac{du}{dx}(x)\right) = f(x) \quad \text{in} \quad]a,b[\quad , \quad u(a) = u_a , \quad u(b) = u_b . \tag{1.5.1.16}$$

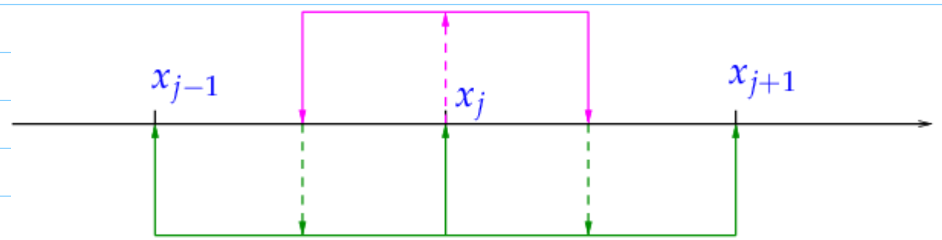
$$\frac{d}{dx}\left(\sigma(x)\frac{du}{dx}(x)\right)_{|x=x_j} \approx \frac{2}{h_j + h_{j+1}}\left(\sigma(x_{j+1/2})\frac{du}{dx}(x_{j+1/2}) - \sigma(x_{j-1/2})\frac{du}{dx}(x_{j-1/2})\right) .$$

$x_{j+1/2} := \frac{1}{2}(x_j + x_{j+1})$

$\hookrightarrow$ two-sided difference quotient

$$\frac{du}{dx}(x_{j+1/2}) \approx \frac{u(x_{j+1}) - u(x_j)}{h_{j+1}} .$$

$$-\frac{d}{dx}\left(\sigma(x)\frac{du}{dx}(x)\right)_{|x=x_j} \approx \frac{\sigma(x_{j-1/2})\dfrac{u(x_j)-u(x_{j-1})}{h_j} - \sigma(x_{j+1/2})\dfrac{u(x_{j+1})-u(x_j)}{h_{j+1}}}{\frac{1}{2}(h_j + h_{j+1})} . \tag{4.1.1.9}$$



Construction of (4.1.1.9):

magenta: outer difference quotient

green: inner difference quotients

[requires $u(x)$ only at gridpoints]

$\Rightarrow$ Unknown of FDM : $\mu_j \approx u(x_j)$, $j = 0, \dots, M+1$

(4.1.1.9) + $\frac{d}{dx}\left(\sigma(x)\frac{du}{dx}\right) = f$:

$$-\frac{\sigma(x_{j+1/2})}{\frac{1}{2}h_{j+1}(h_j + h_{j+1})}\mu_{j+1} + \frac{h_{j+1}^{-1}\sigma(x_{j+1/2}) + h_j^{-1}\sigma(x_{j-1/2})}{\frac{1}{2}(h_j + h_{j+1})}\mu_j - \frac{\sigma(x_{j-1/2})}{\frac{1}{2}h_j(h_j + h_{j+1})}\mu_{j-1} = f(x_j) \tag{4.1.1.14}$$

for every interior gridpoint, $j = 1, \dots, M-1$

$\hat{=}$ $(M-1) \times (M+1)$ LSE for $\vec{\mu} = [\mu_0, \dots, \mu_M]^T \in \mathbb{R}^{M+1}$

$$\mathbf{A}\vec{\mu} = \vec{\varphi}\,,\quad \text{with}\quad (\mathbf{A})_{j,\ell} := \begin{cases} 0 & \text{, if } |j-l|>1\,,\\ -\dfrac{\sigma(x_{j+1/2})}{\frac{1}{2}h_{j+1}(h_j+h_{j+1})} & \text{, if } \ell = j+1\,,\\ \dfrac{h_{j+1}^{-1}\sigma(x_{j+1/2})+h_j^{-1}\sigma(x_{j-1/2})}{\frac{1}{2}(h_j+h_{j+1})} & \text{, if } j = l\,,\\ -\dfrac{\sigma(x_{j-1/2})}{\frac{1}{2}h_j(h_j+h_{j+1})} & \text{, if } \ell = j-1\,,\end{cases} \tag{4.1.1.15}$$

$$\varphi_j := f(x_j)\,,$$

$\downarrow$ tridiagonal

For $h_j = h$ (equidistant mesh)

$$\mathbf{A} = \frac{1}{h^2}\begin{bmatrix} -\sigma_{\frac{1}{2}} & \sigma_{\frac{1}{2}}+\sigma_{\frac{3}{2}} & -\sigma_{\frac{3}{2}} & & & & \\ & -\sigma_{\frac{3}{2}} & \sigma_{\frac{3}{2}}+\sigma_{\frac{5}{2}} & -\sigma_{\frac{5}{2}} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & -\sigma_{M-\frac{3}{2}} \\ & & & & -\sigma_{M-\frac{3}{2}} & \sigma_{M-\frac{1}{2}}+\sigma_{M-\frac{3}{2}} & -\sigma_{M-\frac{1}{2}} \end{bmatrix}, \tag{4.1.1.16}$$

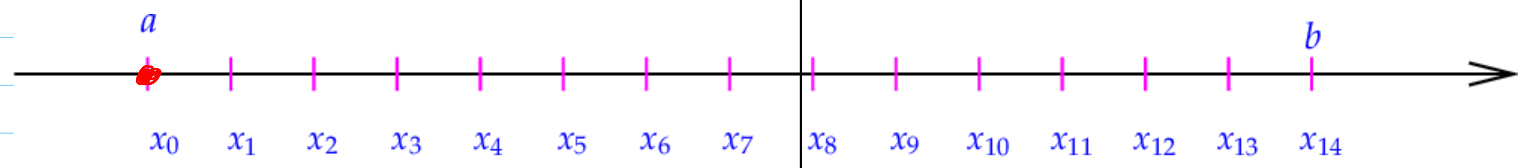
with : $\sigma_{1/2} = \sigma(x_{1/2})$: $M-1$ equ. for $M+1$ unknowns

§ 4.1.1.17 : FDM for Dirichlet 2-pt. BVP

Known : $\mu_0 = u_a$, $\mu_n = u_b$ $\Rightarrow$ $M-1$ unknowns

§ 4.1.1.19 : FDM for Neumann 2-pt BVP

$$-\frac{d}{dx}\left(\sigma(x)\frac{du}{dx}(x)\right) = f(x) \quad \text{in} \quad]a,b[\quad , \quad -\sigma(a)\frac{du}{dx}(a) = v_a\,, \quad \sigma(b)\frac{du}{dx}(b) = v_b\,, \tag{4.1.1.20}$$



1-sided DQ $\qquad -\sigma(a)\dfrac{\mu_1 - \mu_0}{h} = v_a$

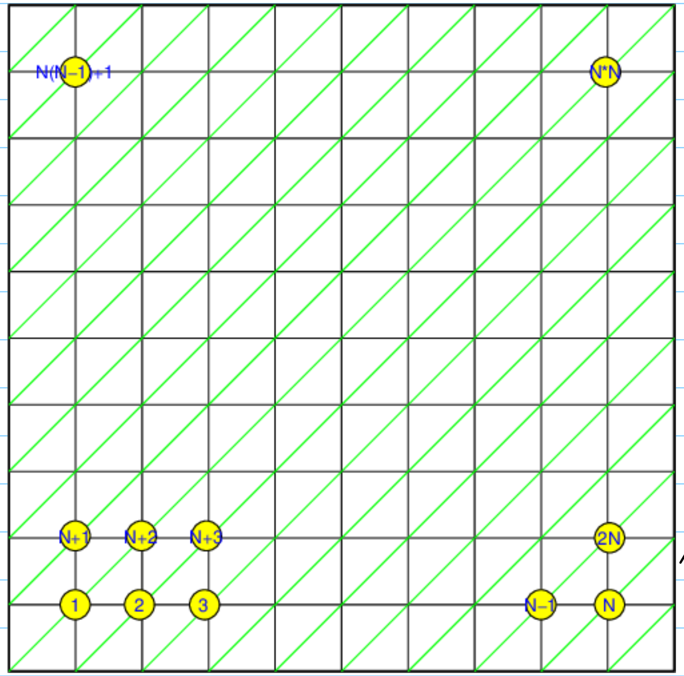
Neumann b.c. $\Rightarrow$ two extra equations

(4)

4.1.2. FDM in Two Dimensions

§ 4.1.2.1. Model problem and structured mesh

$$-\Delta u = -\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = f \quad \text{in } \Omega :=]0,1[^2 \quad , \quad u = 0 \quad \text{on } \partial\Omega \,. \tag{4.1.2.2}$$



◁ An equidistant tensor-product grid ("a structured mesh")

$h = \frac{1}{N+1}$

Symmetric directional difference-quotient approximation of partial derivatives:

$$\begin{aligned}
\frac{\partial^2}{\partial x_1^2} u_{|\mathbf{x}=(\xi,\eta)} &\approx \frac{u(\xi-h,\eta) - 2u(\xi,\eta) + u(\xi+h,\eta)}{h^2} \,, \\
\frac{\partial^2}{\partial x_2^2} u_{|\mathbf{x}=(\xi,\eta)} &\approx \frac{u(\xi,\eta-h) - 2u(\xi,\eta) + u(\xi,\eta+h)}{h^2} \,.
\end{aligned} \tag{4.1.2.4}$$

▶ $-\Delta u_{|\mathbf{x}=(\xi,\eta)} \approx \frac{1}{h^2}(4u(\xi,\eta) - u(\xi-h,\eta) - u(\xi+h,\eta) - u(\xi,\eta-h) - u(\xi,\eta+h)) \,.$

Unknowns : $\mu_{i,j} \approx u(ih, jh)$, $1 \leq i,j \leq N$

$-\Delta u = f$

$\Downarrow$

$$\frac{1}{h^2}(4u(ih,jh) - u(ih-h,jh) - u(ih+h,jh) - u(ih,jh-h) - u(ih,jh+h)) = f(ih,jh) \,,$$

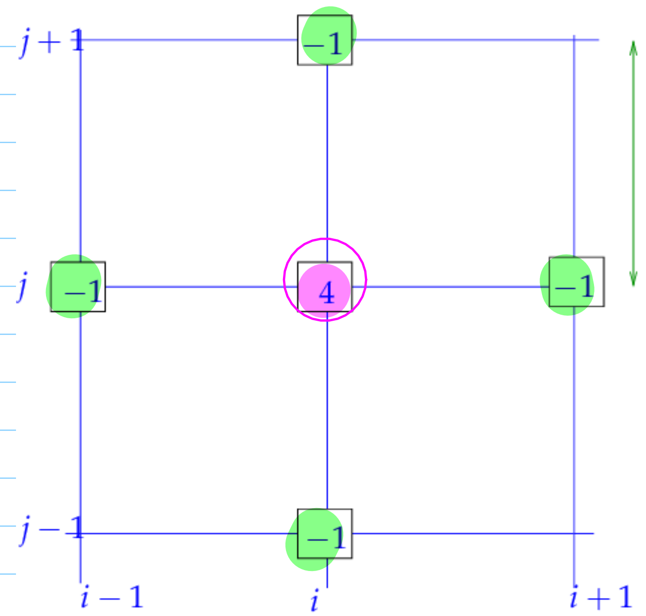
$$\frac{1}{h^2}(4\mu_{i,j} - \mu_{i-1,j} - \mu_{i+1,j} - \mu_{i,j-1} - \mu_{i,j+1}) = f(ih,jh) \,. \tag{4.1.2.6}$$

$\Rightarrow$ $N^2 \times N^2$ - LSE

§ 4.1.2.10. Stencil notation

Unknowns "located at" grid points $\Rightarrow$ visualization of (4.1.2.6)

(4.1.2.6) $\Rightarrow$ $\frac{1}{h^2}$ ·



← stencil anchored at $(ih, jh) \in \Omega$.

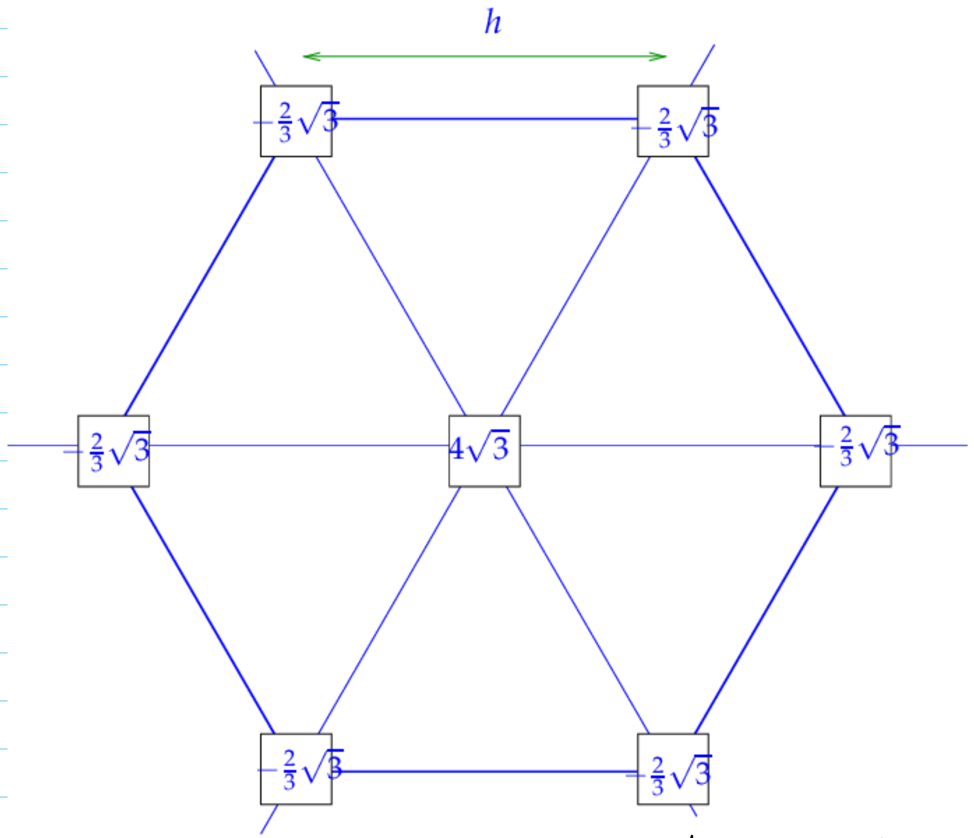
stencil notation:

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_h$$

Translation-invariant PDE and grid

⇒ LSE described by a single stencil

Rem 4.1.2.11 (Stencils on more general meshes)



$$\frac{2}{3}\sqrt{3}\cdot\begin{bmatrix} & -1 & & -1 & \\ -1 & & 6 & & -1 \\ & -1 & & -1 & \end{bmatrix}_h$$

Equilateral triangular mesh

4.1.3. Finite Differences and Finite Elements

§ 4.1.3.1. FDM & FEM for 2-pt. BVPs

Sect. 4.1.1.1 Sect. 2.3. (p.w. linear FE + numerical quadrature)

⇒ Same LSE !

§ 4.1.3.3. FDM vs. FEM in 2D

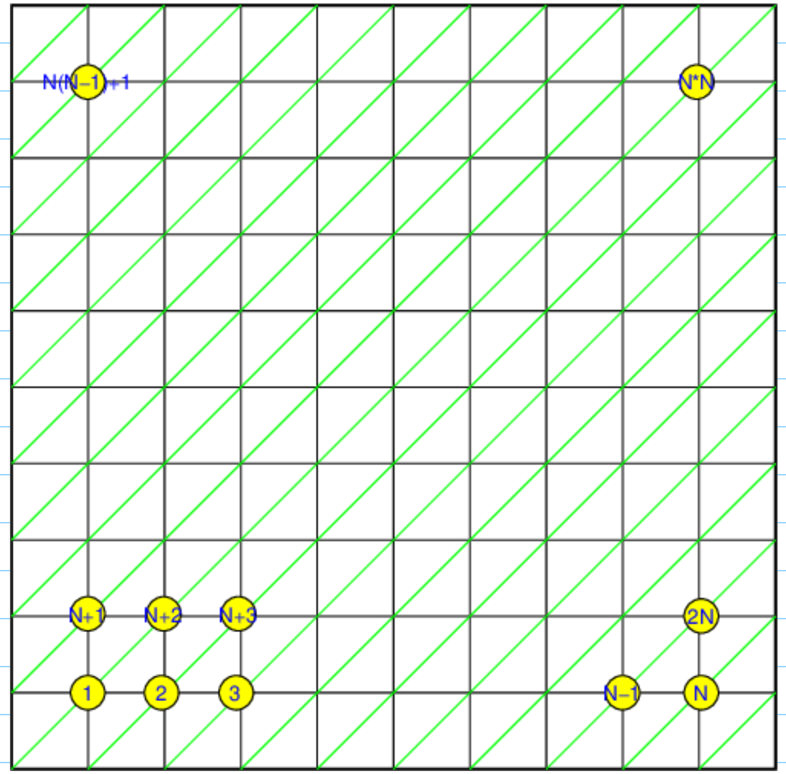
→ Setting of Sect. 4.1.2

$-\Delta u = f$ in Ω

$u = 0$ on $\partial\Omega$

$\Downarrow$

Same LSE from FDM and p.w. Lagrangian FEM



(Most) finite difference schemes	$\leftrightarrow$	finite element Galerkin schemes with numerical quadrature on structured meshes

efficient implementation possible

FDM have difficulties handling complex geometries.

▷ Review questions 4.1.3.5