ETH Lecture 401-0674-00L Numerical Methods for (Partial) Differential Equations

Course Video
Section 4.2: Finite-Volume Methods (FVMs)

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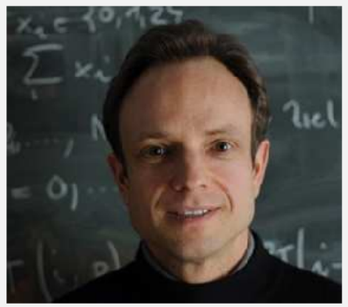


Dependencies. [Lecture → Section 1.6], [Lecture → Section 2.5.1], [Lecture → Section 2.4]

Duration: 42 minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

IV. Beyond FEM
Alternative Discretizations
4.2. Finite-Volume Methods (FVM)

Sect. 1.6 :

Balance law * $\int_{\partial V} \mathbf{j} \cdot \mathbf{n}\, dS = \int_V f(x)\, dx \quad \forall$ volumes V

+

Flux law $\mathbf{j} = -\kappa(x) \operatorname{grad} u$

$\triangleright \quad -\operatorname{div}(\kappa \operatorname{grad} u) = f \text{ in } \Omega$

+ Dirichlet b.c. $u = g$ on $\partial\Omega$

* in integral form

4.2.1 Discrete Balance Laws

$$\int_{\partial V} \mathbf{j}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_V f\,\mathrm{d}\boldsymbol{x} \quad \text{for all "control volumes" } V\,, \tag{1.6.0.3}$$

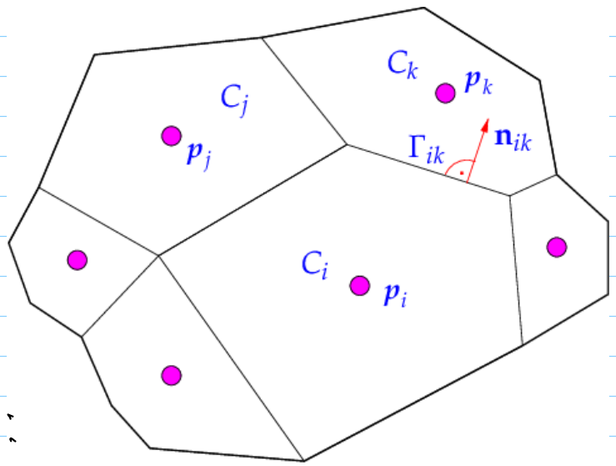
Idea of FMV:

Enforce the conservation/balance law (1.6.0.3) only for finitely many **control volumes**.

► First ingredient of FVM: Finitely many *non-overlapping* **control volumes** C_i, $i = 1,\ldots,\tilde{M}$, covering the computational domain: $\overline{\Omega} = \bigcup_{i=1}^{\tilde{M}} \overline{C}_i$

2D: control polygons

$T_{ik} := \partial C_i \cap \partial C_k$



To every C_i we assign an unknown $\mu_i \in \mathbb{R}$:

$\mu_i \approx u(p_i)$, $p_i \in C_i$ $\qquad$ $\mu_i \approx \frac{1}{|C_i|}\int_{C_i} u(x)\,dx$

§4.2.1.5 From discrete flux to systems of equations

► Second ingredient of FVM local **numerical fluxes**

For two adjacent cells C_k, C_i with common edge $\Gamma_{ik} := \overline{C}_i \cap \overline{C}_k$.

Numerical flux $J_{ik} = \Psi(\mu_i,\mu_k) \approx \int_{\Gamma_{ik}} \mathbf{j}\cdot\boldsymbol{n}_{ik}\,\mathrm{d}S$

$\Psi \hat{=}$ numerical flux function, $\mathbf{j}$ = (heat) flux, see (1.6.0.2), $\boldsymbol{n}_{ik} \hat{=}$ edge normal, see Fig. 264.

Notation: $\mathcal{M}_i := \{j : \overline{C}_j \cap \overline{C}_i \neq \emptyset\}$

$$\int_{\partial C_i} \mathbf{j}\cdot\mathbf{n}_i\,\mathrm{d}S = \int_{C_i} f\,\mathrm{d}\boldsymbol{x} \quad\Rightarrow\quad \sum_{k\in\mathcal{M}_i} J_{ik} = \int_{C_i} f\,\mathrm{d}\boldsymbol{x}\,. \tag{4.2.1.6}$$

$\partial C_i = \cup\{T_{ik}, k \in \mathcal{M}_i\}$

$\Downarrow\ J_{ik} = \Psi(\mu_i,\mu_k)$

$$\sum_{k\in\mathcal{U}_i} \Psi(\mu_i,\mu_k) = \int_{C_i} f\,\mathrm{d}\boldsymbol{x} \quad \forall i = 1,\ldots,\tilde{M}\,. \tag{4.2.1.7}$$

$\hat{=}$ $\tilde{M}$ eqns. for $\tilde{M}$ unknowns μ_i

1-pt. numerical quadrature $\approx |C_i| f(p_i)$

4.2.2. Dual Meshes

§ 4.2.2.1 (Voronoi dual meshes)

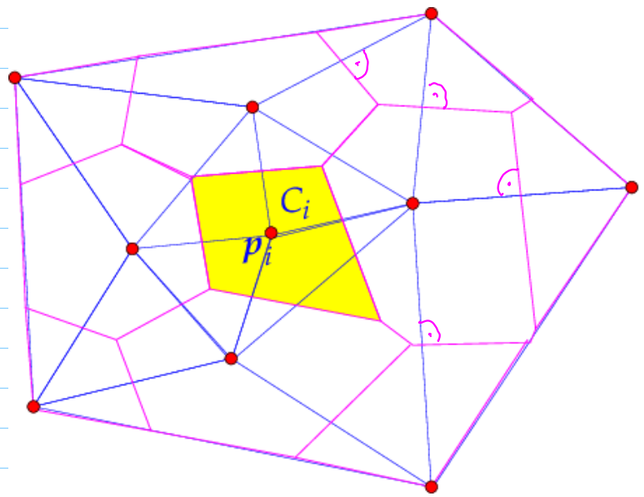
$\mathcal{M}$ ≙ (primal) triangular mesh $\mathcal{M}$ of $\Omega \subset \mathbb{R}^2$ with node set $\{p_1, \ldots, p_M\}$

Voronoi cell for p_i

$$C_i := \{\mathbf{x} \in \Omega : \|\mathbf{x} - \boldsymbol{p}_i\| < \|\mathbf{x} - \boldsymbol{p}_j\| \; \forall j \neq i\} . \tag{4.2.2.2}$$

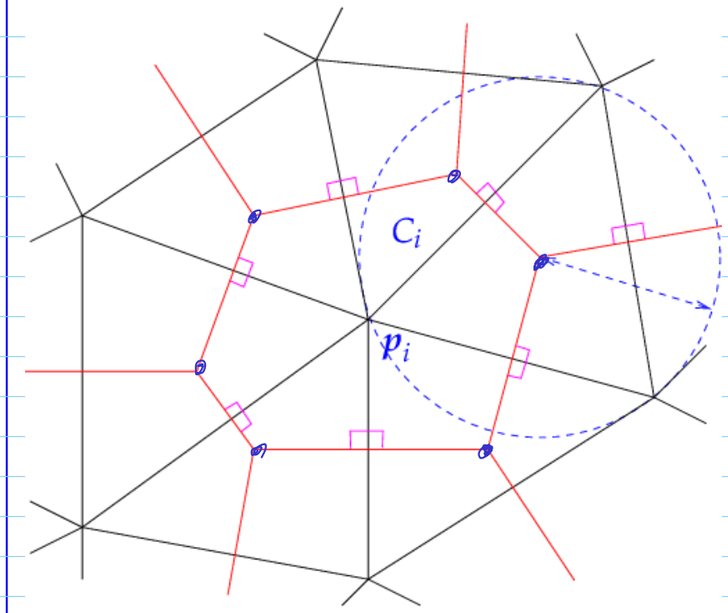
↳ points closer to p_i than to any other node

$\{C_i\}$ ≙ **Voronoi dual mesh**



◁ Voronoi dual mesh:

- ●: nodes of primal mesh
- —: edges of primal mesh
- —: edges Voronoi dual mesh
- ■: Voronoi dual cell for $\boldsymbol{p}_i$

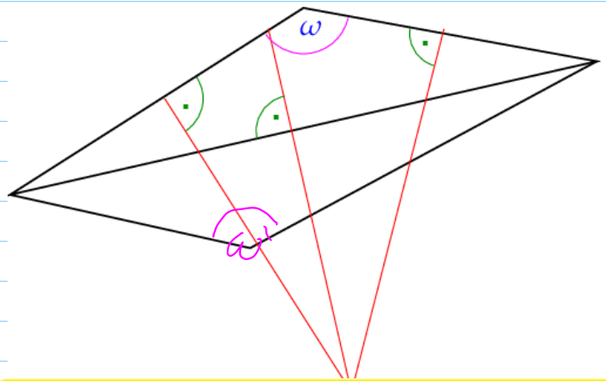


Construction

Connect circumcenters of triangles adjacent to p_i ⇒ ∂C_i

▷ primal and dual edges are orthogonal

Rem 4.2.2.3 (Geometric obstruction)



⇐ Obtuse angle ω:

- ➢ circumcenter ∉ triangle
- ➢ $\overline{C}_i \cap \overline{C}_j \neq \emptyset \not\Rightarrow$ nodes i, j connected by edge
- ➢ geometric construction breaks down
- ➢ connectivity of unknowns unclear

Theorem 4.2.2.4. Angle condition for Voronoi dual meshes

The following angle condition ensures that the Voronoi cells belonging to adjacent nodes of a triangular mesh have a common edge ($\overline{C}_i \cap \overline{C}_j \neq \emptyset$ ⇔ nodes i, j connected by edge of $\mathcal{M}$):

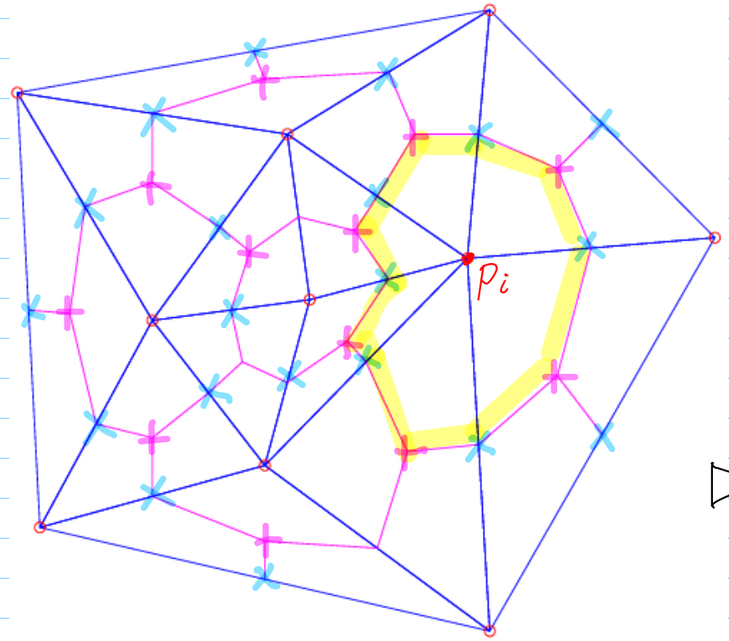
(i) sum of angles facing interior edge $\leq \pi$,

(ii) angles facing boundary edges $\leq \pi/2$.

Definition 4.2.2.5. Delaunay triangulation

Triangular meshes satisfying the angle conditions (i) and (ii) from Thm. 4.2.2.4 are called Delaunay triangulations.

§ 4.2.2.6. Barycentric dual meshes



Construction

Connect the barycenters of triangles with midpoint of edges adjacent to p_i

▷ always possible

For dual meshes

control volumes = dual cells $\leftrightarrow$ primal nodes

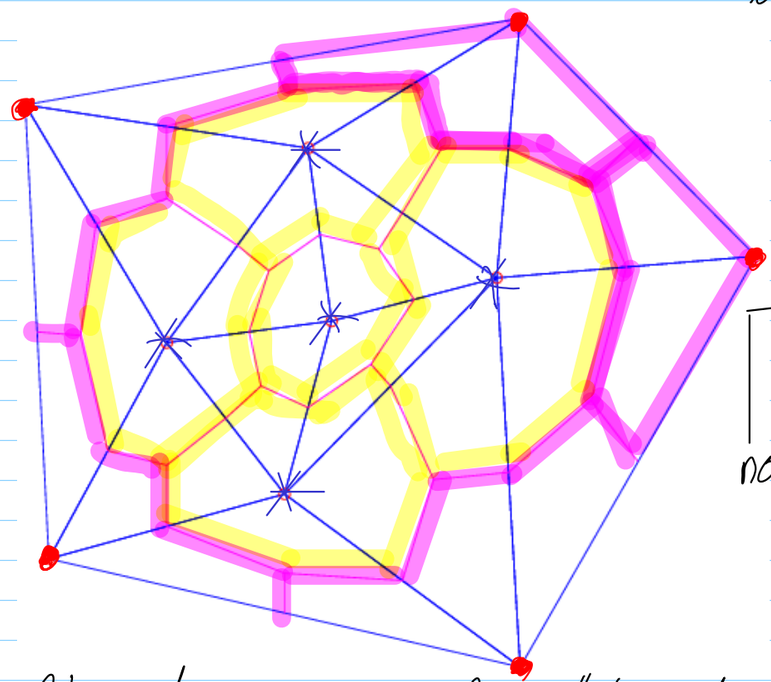
Rem 4.2.2.7 : FVM : Incorporation of Dirichlet boundary conditions

$u = g$ on $\partial\Omega$ (continuous)

$u(\bullet)$ is known

$\rightarrow p_i := g(p_i)$ if $p_i \in \partial\Omega$

no longer unknown



Also drop equs. for "boundary control vols"

active dual. vols (▬) = # equs.
= # interior primal nodes
= # unknowns

(5)

4.2.3. Relationship of FVM and FEM

Setting:
- 2D triangular primal mesh $\mathcal{M}$ of polygonal $\Omega \subset \mathbb{R}^2$ [Delaunay triangulation]
- Control volumes from Voronoi dual mesh
- $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ (4.2.3.1)

§ 4.2.3.2 Numerical flux by interpolation

Idea: Obtain the numerical flux from Fourier's law

$$\mathbf{j}(\boldsymbol{x}) = -\kappa(\boldsymbol{x})\,\mathbf{grad}\,u(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \Omega\,, \tag{1.6.0.5}$$

applied to a (sufficiently smooth) $u_h : \Omega \mapsto \mathbb{R}$ *reconstructed* from dual cell values μ_i.

$u_h \hat{=}$ p.w. linear interpolant of μ_i on $\mathcal{M}$

$$u_h = I_1 \vec{\mu} := \sum_{i=1}^{N} \mu_i b_h^i \in S_1^0(\mathcal{M})$$

$N = \#\mathcal{V}(\mathcal{M})$; b_h^i = tent basis function

▷ $I_{ik} = -\int_{T_{ik}} \mathrm{grad}\, I_1\vec{\mu} \cdot \mathbf{n}_{ik}\, dS$ [$k \equiv I$]

▷ FVM equation for C_i:

$$\sum_{k \in \mathcal{U}_i} -\int_{T_{ik}} \mathrm{grad}\, I_1\vec{\mu} \cdot \mathbf{n}_{ik}\, dS = \int_{C_i} f(x)\, dx$$

linear equ. $\leftrightarrow$ i-th row of FV-LSE

▷
$$\sum_{k\in\mathcal{U}_i} \int_{\Gamma_{ik}} \mathbf{grad}\, l_1\vec{\mu} \cdot \mathbf{n}_{ik}\, dS = \mu_i \underbrace{\int_{\partial C_i} \mathbf{grad}\, b_h^i\, dS}_{= \text{matrix entry } -(\mathbf{A})_{ii}} + \sum_{j\in\mathcal{U}_i} \mu_j \underbrace{\left(\sum_{k\in\mathcal{U}_i} \int_{\Gamma_{ik}} \mathbf{grad}\, b_h^j \cdot \mathbf{n}_{ik}\, dS\right)}_{= \text{matrix entry } -(\mathbf{A})_{ij}}$$

$$= -\int_{C_i} f(\boldsymbol{x})\, d\boldsymbol{x}\,.$$

$$\Rightarrow \quad (\mathbf{A})_{ij} = -\int_{\partial C_i} \mathbf{grad}\, b_h^j \cdot \mathbf{n}_i\, dS\,, \quad i,j \in \{1,\ldots,N\}\,. \tag{4.2.3.5}$$

where $\boldsymbol{n}_i \hat{=}$ exterior unit normal vector to ∂C_i.

We can compute: A = FE Galerkin matrix for $S_{1,0}^0(\mathcal{M})$-FEM for $-\Delta u = f$

The finite volume discretization and the finite element Galerkin discretization spawn the same system matrix for the model problem (4.2.3.1).

4.2.60_eq:vacmsp_p

⑥ Then why FVM ?

▷ Physics aware discretization policy for balance laws beyond 2nd-order elliptic BVPs (non-linear "conservation laws")

$$\operatorname{div}(\alpha \operatorname{grad} u + g(x,u)) = f \quad \text{in } \Omega \subset \mathbb{R}^d$$

$$g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^d$$

▷ Review questions Q 4.2.3.A