ETH Lecture 401-0674-00L Numerical Methods for (Partial) Differential Equations

Course Video
Section 4.4: Collocation Methods

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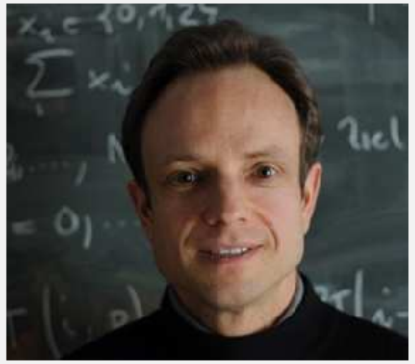
Date: March 15, 2022

(C) Seminar für Angewandte Mathematik, ETH Zürich



Prerequisites.

- Familiarity with polynomial Lagrangian interpolation
- Knowledge about cubic splines

Dependencies. [Lecture → Section 4.3], [Lecture → Section 1.5.3]

Duration: 30 minutes

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

IV. Beyond FEM
Alternative Discretizations

4.4. Collocation Methods

Focus on classical view of (P)DEs / BVPs

$$-\operatorname{div}(\alpha(x)\,\mathbf{grad}\,u) = f \text{ in } \Omega \;,\quad u = g \text{ on } \partial\Omega \tag{4.4.0.2}$$

- $f, g \in C^0$, $\alpha \in C^1$
- $u \in C^2(\Omega)$

$$\mathsf{L}u = f \text{ in } C^0(\Omega) \;,\quad \mathsf{B}u = g \text{ in } C^0(\partial\Omega)\,, \tag{4.4.0.4}$$

↳ diff. op. ↳ boundary op. : both linear

Idea of collocation discretization

❶ Seek an approximate solution u_h of (4.4.0.4) in a finite-dimensional trial space

$$V_{0,h} \subset C^2(\Omega) \quad , \quad N := \dim V_{0,h} < \infty \, . \tag{4.4.0.6}$$

❷ Pick finitely many collocation nodes/points

$$x_1, \dots, x_n \in \Omega \quad , \quad y_1, \dots, y_m \in \partial\Omega \, , \quad m, n \in \mathbb{N} \, .$$

Impose collocation conditions:

$$u_h \in V_{0,h}: \quad \begin{array}{l} (\mathsf{L}u_h)(x_j) = f(x_j) \, , \quad j = 1, \dots, n \, , \\ (\mathsf{B}u_h)(y_\ell) = g(y_\ell) \, , \quad j = 1, \dots, m \, . \end{array} \tag{4.4.0.7}$$

u_h is to solve BVP a finitely many point

§ 4.4.0.8 : Choice of collocation points

The trial space $V_{0,h}$ and the collocation nodes x_j, $j = 1, \dots, n$, and y_ℓ, $\ell = 1, \dots, m$ have to satisfy: For all $f_1, \dots, f_n \in \mathbb{R}$, $g_1, \dots, g_m \in \mathbb{R}$ the interpolation problem

$$u_h \in V_{0,h}: \quad u(x_j) = f_j, \; j = 1 \dots, n \quad u(y_\ell) = g_\ell, \; \ell = 1, \dots, m \, , \tag{4.4.0.9}$$

has a unique solution.

$\Rightarrow \quad N = m + n$

§ 4.4.0.11 : From collocation conditions to systems of equations

Collocation method: second step

Choose an ordered basis $\mathfrak{B}_h = \{b_h^1, \dots, b_h^N\}$ of $V_{0,h}$ and plug the basis representation

$$u_h = \mu_1 b_h^1 + \dots + \mu_N b_h^N \, , \quad \mu_1, \dots, \mu_N \in \mathbb{R} \, , \tag{4.4.0.13}$$

into the collocation conditions (4.4.0.7), yielding the collocation equations

$$\mu_1 (\mathsf{L}b_h^1)(x_j) + \dots + \mu_N (\mathsf{L}b_h^N)(x_j) = f(x_j) \, , \quad j = 1, \dots, n \, , \tag{4.4.0.14a}$$

$$\mu_1 (\mathsf{B}b_h^1)(y_\ell) + \dots + \mu_N (\mathsf{B}b_h^N)(y_\ell) = g(y_\ell) \, , \quad \ell = 1, \dots, m \, . \tag{4.4.0.14b}$$

require linearity of L & B !

4.4.1. Spectral Collocation Method

$\triangleq$ $V_{0,h}$ as for spectral Galerkin (Sect. 4.3)

Ex. 4.4.1.1 : Spectral collocation on a square

$$-\Delta u = f \quad \text{in} \quad \Omega :=]-1,1[^2 \quad , \quad u = g \quad \text{on} \quad \partial\Omega \, . \tag{4.4.1.2}$$

$f, g \in C^0$

3 • $V_{0,h} = Q_p(\mathbb{R}^2)_{|\Omega} \Rightarrow N = (p+1)^2$

• Basis : tensor-product Legendre polynomials

$$\mathfrak{B}_h := \{\boldsymbol{x} = [x_1, x_2] \mapsto P_r(x_1)P_s(x_2) : r,s \in \{0,\dots,p\}\}\,, \qquad (4.4.1.3)$$

$\longrightarrow$ "well-conditioned"

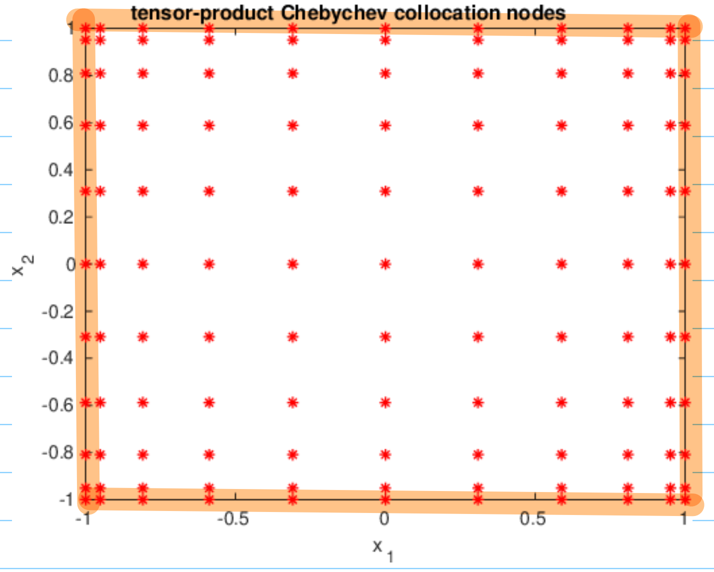
• Collocation nodes : tensor-product construction

[Start from good interpolation nodes]

$$\xi_j := \cos(\pi/p\,j) \in [-1,1]\,, \quad j = 0,\dots,p\,. \qquad (4.4.1.5)$$

$\hookrightarrow$ dilated Chebychev nodes

$$\underbrace{\{\boldsymbol{x}_j\}_{j=1}^{n}}_{\in\Omega} \cup \underbrace{\{\boldsymbol{y}_\ell\}_{\ell=1}^{m}}_{\in\partial\Omega} = \left\{[\xi_r, \xi_s]^\top \in \overline{\Omega} : r,s \in \{0,\dots,p\}\right\}. \qquad (4.4.1.6)$$



▷ Collocation LSE (dense)

$L\{\boldsymbol{x} \to P_r(x_1)P_s(x_2)\}$

$$-\sum_{r,s=0}^{p} \mu_{r,s}\left(P_r''(\xi_k)P_s(\xi_j) + P_r(\xi_k)P_s''(\xi_j)\right) = f\left(\begin{bmatrix}\xi_k \\ \xi_j\end{bmatrix}\right), \quad k,j \in \{1,\dots,p-1\}\,, \qquad (4.4.1.7)$$

$$(Bu_h)(\boldsymbol{y}_\ell) = g(\boldsymbol{y}_\ell) \iff \sum_{r,s=0}^{p} \mu_{r,s} P_r(y_{\ell,1})P_s(y_{\ell,2}) = g(\boldsymbol{y}_\ell), \quad \boldsymbol{y}_\ell = \begin{bmatrix} y_{\ell,1} \\ y_{\ell,2}\end{bmatrix}, \ \ell = 1,\dots,m\,.$$

$\mu_{r,s} \hat{=}$ unknowns : basis expansion coefficients

$$u_h(\boldsymbol{x}) = \sum_{r,s=0}^{p} \mu_{r,s} P_r(x_1)P_s(x_2)\,, \quad \boldsymbol{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \overline{\Omega}\,. \qquad (4.4.1.4)$$

4.4.2 Spline Collocation Method

Recall requirement : $V_{0,h} \subset C^2(\overline{\Omega})$

Definition 4.4.2.1. Cubic spline $S_{3,\mathcal{T}}$

A function $s : [a,b] \mapsto \mathbb{R}$ is a cubic spline function w.r.t. the ordered knot set $\mathcal{T} := \{a = x_0 < x_1 < x_2 < \dots < x_{M-1} < x_M = b\}$, if

(i) $s \in C^2([a,b])$ (twice continuously differentiable),

(ii) $s_{|]x_{j-1},x_j[} \in \mathcal{P}_3(\mathbb{R})$ (a piecewise cubic polynomial)

$d = 1$:

$$V_{0,h} = \{v_h \in S_{3,\mathcal{T}} : v_h''(a) = v''(b) = 0\}$$

- Collocation nodes = knots

$\longrightarrow$ natural cubic spline interpolation

Benefit: Sparse collocation LSE

§ 1.4.2.4: Assessment: collocation methods

Compared to FEM:

✦ Numerical quadrature not required

✦ Implementations confined to *simple domains*

✦ Weaker theoretical guarantees