ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 5.1: Non-Linear Elastic Membrane Models

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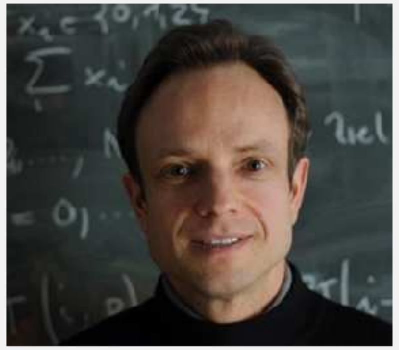


Prerequisites.

- Elementary calculus
- Taylor expansion
- Riemann integral

Dependencies. [Lecture → 1.2.1], [Lecture → Section 1.5]

Duration: 54 minutes

Video and accompanying tablet notes may not match completely!

[Corrections and updates may have been made in tablet notes.]

V Non-Linear Elliptic BVPs

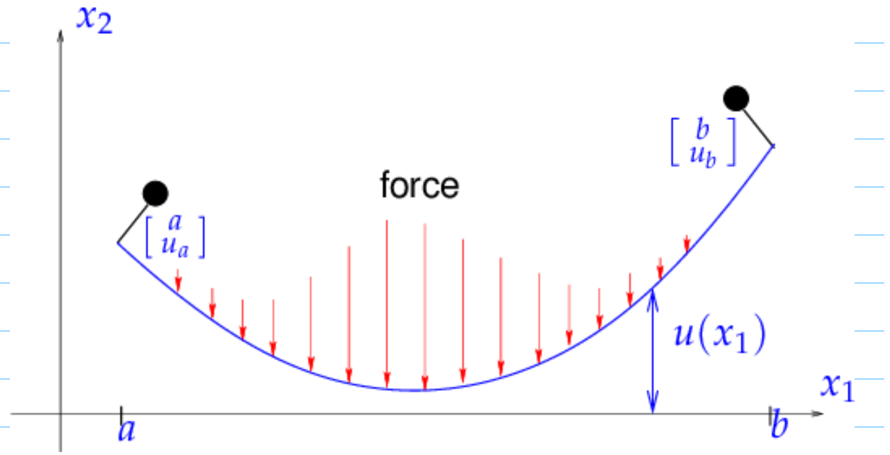
5.1 Non-Linear Elastic Membrane Models

5.1.1 Thin Elastic String Model

5.1.1.1 Recalled: Configuration Space

→ Sect. 1.2.1.

Graph model for the shape of a string



Basic configuration space for elastic string model

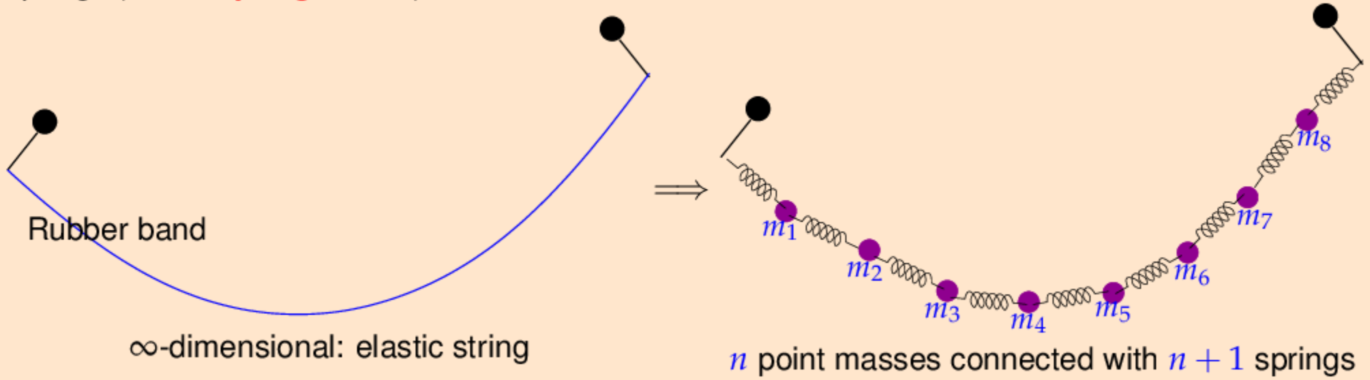
The configuration space for the elastic string model under vertical loading is the infinite-dimensional affine function space

$$\hat{V}_S := \left\{ u \in C^0([a,b]),\ u(a) = u_a,\ u(b) = u_b \right\} . \tag{1.2.1.7}$$

② 5.1.1.2 Mass-Spring Model (MSM)

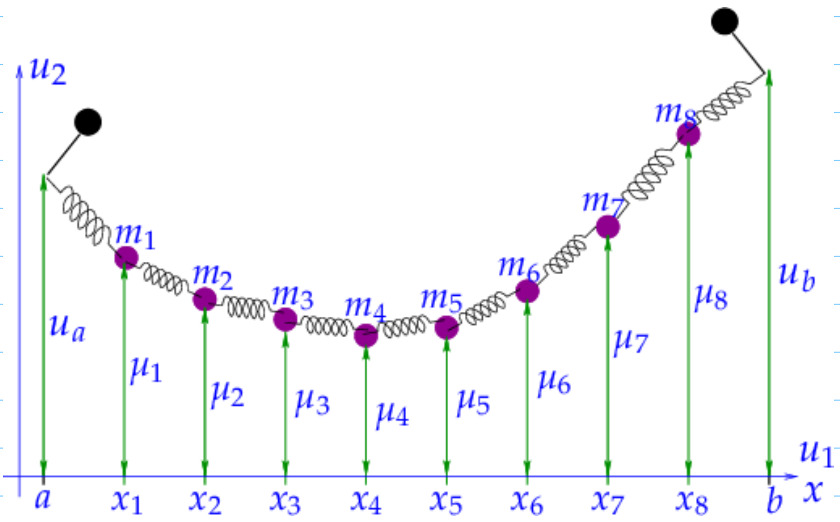
Reduction to finite-dimensional configuration space

The idea is to *approximate* an elastic string as a system of finitely many point masses connected by springs (**mass-spring model**).



Simplified model with n d.o.f. μ_i ≙ vertical displacements of point masses

Equidistant mesh →



Potential energy = $\underbrace{\text{elastic energy } E_{el}}_{\sum_{i=0}^{n} \text{el. en. of } i\text{-th spring}}$ + $\underbrace{\text{gravitational energy}}_{=\sum_{i=1}^{n} g m_i \mu_i}$

[m_i ≙ i-th mass]

Hooke's law : el. en(spring) $= \frac{1}{2}\frac{\sigma}{l_0}(l-l_0)^2$

length of spring

σ ≙ stiffness

$$E_S^{(n)}(\vec{\mu}) := E_{\mathrm{el}}^{(n)}(\vec{\mu}) + E_{\mathrm{g}}^{(n)}(\vec{\mu}) = \frac{1}{2}\sum_{i=0}^{n}\frac{\sigma_i}{l_i}\left(\sqrt{h^2+(\mu_{i+1}-\mu_i)^2}-l_i\right)^2 + \sum_{i=1}^{n} m_i\mu_i g\,, \qquad (5.1.1.12)$$

with $\mu_0 := u_a$, $\mu_{n+1} := u_b$ in order to take into account the pinning conditions.

5.1.1.3 Continuum limit

Linking configuration spaces :

$\mu_i = u(x_i^{(n)})$, $x_i^{(n)} = a + ih$, $h = \frac{b-a}{n+1}$

Assume : $u \in C^2([a,b])$

Limit-compatible parameters

- $l_i = \frac{L}{n+1}$, $\sum l_i = L < b-a$ (tense string)
- $\sigma_i = \sigma(x_{i+1/2}^{(n)})$, $\sigma \in C^0([a,b])$, $\sigma > 0$
- $m_i := \int_{x_{i-1/2}^{(n)}}^{x_{i+1/2}^{(n)}} \varrho(x)\,dx$

mass density $\in C^0([a,b])$

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$$E_{el}^{(n)}(u) = \frac{1}{2}\sum_{i=0}^{n}\frac{n+1}{L}\sigma(x_{i+1/2}^{(n)})\left(\sqrt{h^2+(u(x_{i+1}^{(n)})-u(x_i^{(n)}))^2}-\frac{L}{n+1}\right)^2, \tag{5.1.1.24}$$

$$E_{g}^{(n)}(u) = g\cdot\sum_{i=1}^{n}\int_{x_{i-1/2}^{(n)}}^{x_{i+1/2}^{(n)}}\rho(x)\,dx\cdot u(x_i^{(n)})\,. \tag{5.1.1.25}$$

§ 5.1.1.26 $n \longrightarrow \infty$

$$J_g(\mathbf{u}) = \lim_{n\to\infty} E_g^{(n)}(\mathbf{u}) = \lim_{n\to\infty} g\cdot\sum_{i=1}^{n}\int_{x_{i-1/2}^{(n)}}^{x_{i+1/2}^{(n)}}\rho(x)\,dx\cdot u(x_i^{(n)}) = g\int_a^b\rho(x)u(x)\,dx\,. \tag{5.1.1.27}$$

Limit of elastic energies via Taylor expansion

$$\begin{aligned}\sqrt{h^2+(u(x+\tfrac{1}{2}h)-u(x-\tfrac{1}{2}h))^2} &= \sqrt{h^2+(u'(x)h+O(h^2))^2}\\ &= \sqrt{h^2+|u'(x)|^2h^2+O(h^3)}\\ &= h\sqrt{1+|u'(x)|^2}\sqrt{1+O(h)}\\ &= h\sqrt{1+|u'(x)|^2+O(h^2)} \quad \text{for } h\to 0\,,\end{aligned} \tag{5.1.1.29}$$

$x = x_{i+1/2}^{(n)}$

$\sqrt{1+\delta} = 1+\frac{1}{2}\delta+O(\delta^2) \rightarrow$

$h = \frac{b-a}{n+1}$

$$\begin{aligned}E_{el}^{(n)}(u) &= \frac{1}{2}\sum_{i=0}^{n}\frac{n+1}{L}\sigma(x_{i+1/2}^{(n)})\left(\sqrt{h^2+(u(x_{i+1}^{(n)})-u(x_i^{(n)}))^2}-\frac{L}{n+1}\right)^2\\ &= \frac{1}{2}\sum_{i=0}^{n}\frac{n+1}{L}\sigma(x_{i+1/2}^{(n)})\left(h\sqrt{1+|u'(x_{i+1/2}^{(n)})|^2+O(h^2)}-\frac{L}{n+1}\right)^2\\ &= \frac{1}{2}\frac{b-a}{L}\frac{b-a}{n+1}\sum_{i=0}^{n}\sigma(x_{i+1/2}^{(n)})\left(\sqrt{1+|u'(x_{i+1/2}^{(n)})|^2}-\frac{L}{b-a}+O\left(\frac{1}{n}\right)\right)^2.\end{aligned}$$

for $n \to \infty$

+ Riemann sum

$$q\in C^0([0,1]):\quad \lim_{n\to\infty}\frac{b-a}{n+1}\sum_{j=0}^{n}q(x_{i+1/2}^{(n)}) = \int_a^b q(x)\,dx\,, \tag{5.1.1.31}$$

▶ $$J_{el}(u) = \lim_{n\to\infty}E_{el}^{(n)}(u) = \frac{1}{2}\frac{b-a}{L}\int_a^b\sigma(x)\left(\sqrt{1+|u'(x)|^2}-\frac{L}{b-a}\right)^2dx\,. \tag{5.1.1.32}$$

Model: total potential energy of elastic string

$\leq 1+(u'(x))^2$

$$J_S(u) := J_{el}(u)+J_g(u) = \int_a^b\frac{1}{2}\frac{b-a}{L}\sigma(x)\left(\sqrt{1+|u'(x)|^2}-\frac{L}{b-a}\right)^2+g\rho(x)u(x)\,dx\,. \tag{5.1.1.33}$$

$\rightarrow$ Well-defined for $u \in H^1(]a,b[)$, § 5.1.1.34

Equilibrium condition for elastic string model

The shape of an elastic string with uniformly positive stiffness $\sigma \in C^0_{pw}([a,b])$, mass density $\rho \in C^0_{pw}([a,b])$, relaxed length $L < b - a$, and pinned at $\left[\begin{smallmatrix} a \\ u_a \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} b \\ u_b \end{smallmatrix}\right]$, $a < b$, is the graph of the minimizer u_* of J from (5.1.1.33) over the affine space

$$\widehat{V}_S := \left\{ v \in H^1(]a,b[) : v(a) = u_a \text{ and } v(b) = u_b \right\} : \tag{5.1.1.37}$$

$$u_* = \operatorname*{argmin}_{v \in \widehat{V}_S} J_S(v) . \tag{5.1.1.38}$$

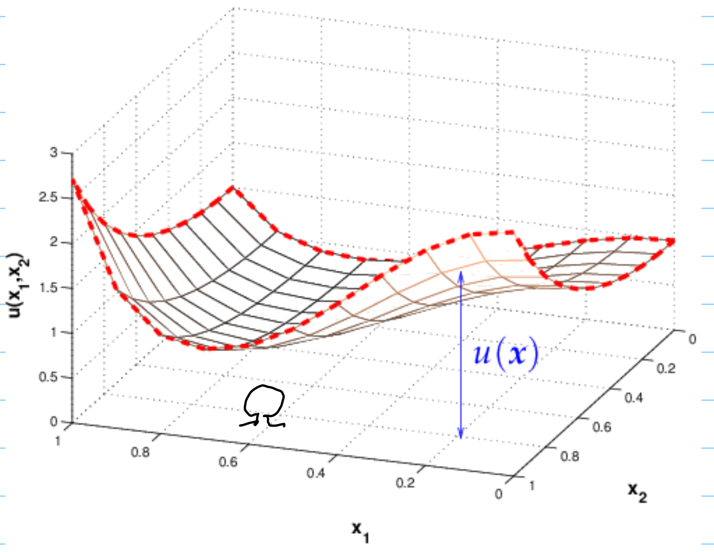
5.1.2. Thin Membrane Model

§ 5.1.2.1. Graph model

Membrane attached to a frame

≙ prescribed displacement over $\partial\Omega$

u ≙ vertical displacement function on $\Omega \subset \mathbb{R}^2$

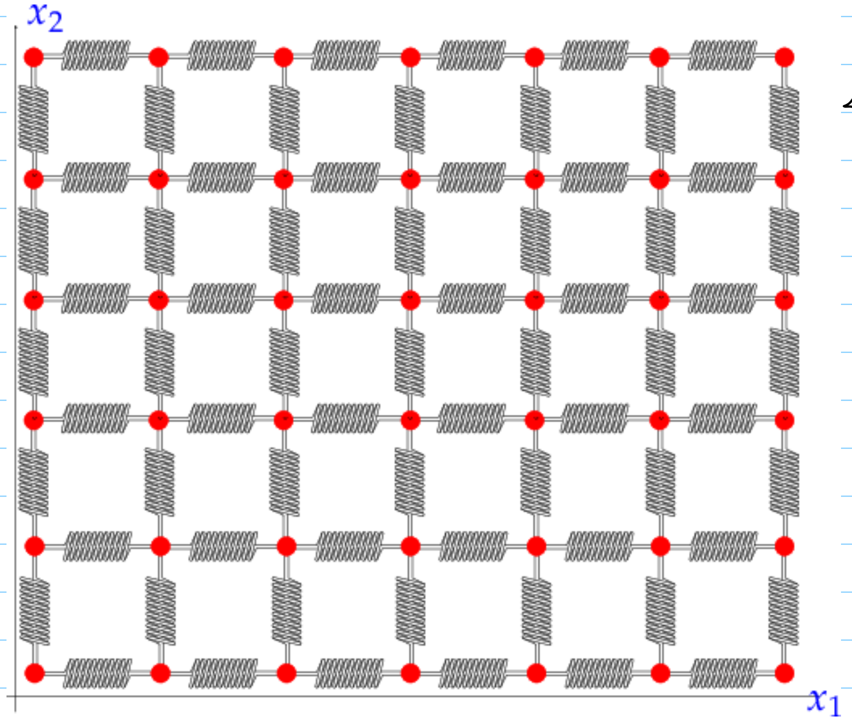


Configuration space for membrane model

Under vertical loading the configuration space for membrane shapes over the base domain $\Omega \subset \mathbb{R}^2$ is the infinite-dimensional affine function space

$$\widehat{V}_M = \left\{ u \in C^0(\overline{\Omega}),\ u|_{\partial\Omega} = d \right\} \tag{1.2.1.12}$$

§ 5.1.2.3 2D MSM



△ $\Omega =]a,b[^2$

- uniform t.p. mesh
- n^2 point masses over mesh nodes

$\mu_{i,j}$ ≙ vertical displacement of mass over

$$x^{(n)}_{i,j} := \begin{bmatrix} a+ih \\ a+jh \end{bmatrix}$$

$$h = \frac{b-a}{n+1}$$

§ 5.1.2.4

▷ Total potential energy (linear springs, relaxed length: $l_0 = \frac{L}{n+1}$)

$$E_M^{(n)}((\mu_{i,j})_{i,j}) = \sum_{j=1}^{n}\sum_{i=0}^{n} \frac{\sigma_{i+1/2,j}}{2l_0}\left(\sqrt{h^2+(\mu_{i+1,j}-\mu_{i,j})^2}-l_0\right)^2 + \sum_{i=1}^{n}\sum_{j=0}^{n} \frac{\sigma_{i,j+1/2}}{2l_0}\left(\sqrt{h^2+(\mu_{i,j+1}-\mu_{i,j})^2}-l_0\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} g m_{i,j}\mu_{i,j}\,. \tag{5.1.2.8}$$

elastic energy ↑ gravitational energy ↑

§ 5.1.2.9 $n \longrightarrow \infty$

- $\mu_{i,j} = u(x_{i,j})$, $u \in C^2(\overline{\Omega})$
- σ via sampling $\sigma : \Omega \to \mathbb{R}^+$
- $m_{i,j}$ by integrals of mass density $\rho : \Omega \to \mathbb{R}$

$$E_M^{(n)}(u) = \frac{1}{2L}\frac{n+1}{n}\Big\{\sum_{j=1}^{n}\sum_{i=0}^{n}\sigma(x_{i+1/2,j})\left(\sqrt{h^2+(u(x_{i+1,j})-u(x_{i,j}))^2}-l_0\right)^2 + \sum_{i=1}^{n}\sum_{j=0}^{n}\sigma(x_{i,j+1/2})\left(\sqrt{h^2+(u(x_{i,j+1})-u(x_{i,j}))^2}-l_0\right)^2\Big\} + \sum_{i=1}^{n}\sum_{j=1}^{n} g \int_{(i-1/2)h}^{(i+1/2)h}\int_{(j-1/2)h}^{(j+1/2)h} \rho([\begin{smallmatrix}x\\y\end{smallmatrix}])\,\mathrm{d}y\mathrm{d}x \cdot u(x_{i,j})\,. \tag{5.1.2.13}$$

$$J_M(u) := \lim_{n\to\infty} E_M^{(n)}(u) = \int_\Omega \frac{1}{2L}\sigma(x)\left(\left(\sqrt{1+\left|\frac{\partial u}{\partial x_1}(x)\right|^2}-\frac{L}{b-a}\right)^2 + \left(\sqrt{1+\left|\frac{\partial u}{\partial x_2}(x)\right|^2}-\frac{L}{b-a}\right)^2\right) + g\,\rho(x)u(x)\,\mathrm{d}x\,. \tag{5.1.2.14}$$

↳ $u \in H^1(\Omega)$ & general Ω admissible

5.1.3. Taut Membrane Limit $\hat{=}$ $L \to 0$

Sect 1.2.1 : Quadratic energies ?

$$J_S(u) := J_{el}(u) + J_g(u) = \int_a^b \frac{1}{2}\frac{b-a}{L}\sigma(x)\left(\sqrt{1+|u'(x)|^2}-\cancel{\frac{L}{b-a}}\right)^2 + g\rho(x)u(x)\,\mathrm{d}x\,. \tag{5.1.1.33}$$

Renormalized stiffness : $\hat{\sigma}(x) = \frac{b-a}{L}\sigma(x)$: independent of L

Set $L := 0$

$$\tilde{J}_S(u) = \frac{1}{2}\int_a^b \hat{\sigma}(x)\left(1+|u'(x)|^2\right) + g\rho(x)u(x)\,\mathrm{d}x\,. \tag{5.1.3.1}$$

↳ A quadratic functional

irrelevant for minimizer

⑥

▷ Please look at Review Questions 5.1.1.42 and 5.1.3.3